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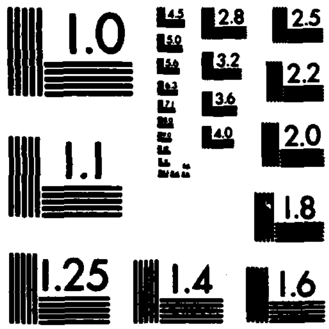

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ABSTRACT

GREAT SPHERE FIBRATIONS OF MANIFOLDS

John Petro, Author

Herman R. Gluck, Supervisor

STATEMENT OF PROBLEM: If E is a smooth closed manifold which is smoothly fibred by k-spheres and smoothly embedded in S^N (the unit N-sphere in $\mathbb{R}^{N+1}$) so that these k-sphere fibres appear as great k-spheres in S^N then we say that E is fibred by great k-spheres. There are three questions that guide our study: (1) given two ~~such~~ g fibrations are they topologically equivalent, (2) is it possible to deform one ~~such~~ g fibration to another through a one parameter family of ~~such~~ g fibrations, and (3) what is the homotopy type of the space of all ~~such~~ g fibrations? We address these questions for great circle fibrations of odd dimensional round spheres and arbitrary great k-sphere fibrations. ~~of $S^m \times S^n \subseteq S^{m+n+1}$.~~

RESULTS: Let M be the base space of a great circle fibration of S^{2n-1}. We show there are two complex structures on $\mathbb{R}^{2n}$ giving embeddings i_{H_1} and i_{H_2} of $\mathbb{C}P^{n-1}$ into $\tilde{G}_2\mathbb{R}^{2n}$ such that there exists a homotopy $g : I \times \mathbb{C}P^{n-1} \longrightarrow \tilde{G}_2\mathbb{R}^{2n}$ with g_0 equal to i_{H_j} for $j = 1$ or 2 and $g_1 = i \cdot h$ where $h : \mathbb{C}P^{n-1} \longrightarrow M$ is a homotopy equivalence and $i : M \longrightarrow \tilde{G}_2\mathbb{R}^{2n}$ is the natural embedding of M.

We next turn to great k-sphere fibrations of

$S^m \times S^n$ and prove a number of general statements concerning the existence or non-existence of great k-sphere fibrations depending on k , m and n. Our other major result is a complete answer to the three questions above for great 3-sphere fibrations of $S^3 \times S^3 \subseteq S^7$.

Finally we give an example of a great 3-sphere fibration of S^7 with no orthogonal pairs of fibres. This is in contrast to the situation for great circle fibrations of S^3.

GREAT SPHERE FIBRATIONS OF MANIFOLDS

John Petro

A DISSERTATION

in

MATHEMATICS

Presented to the faculties of the University of Pennsylvania in Partial Fulfillment of the Requirements for the Degree of Doctor of Philosophy.

1983

Herman R Gluck

Supervisor of Dissertation

Graduate Group Chairman

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My wife Barbara is well deserving of a public expression of love and thanks. She allowed me to go through graduate school and at the same time experience the joys of having a family. She helped make the stress and pressure bearable.

Finally I thank the United States Air Force which made my three years at the University of Pennsylvania possible. I am proud to be a member of this organization.

TABLE OF CONTENTS

SECTION I

Let E be a smooth closed n-manifold which

a) smoothly fibred by k-spheres, and

b) smoothly embedded in S^N (the unit N-sphere in $\mathbb{R}^{N+1}$) so that these k-sphere fibres appear as <u>great</u> k-spheres in S^N.

We say simply that E is <u>fibred by great k-spheres</u>.

Such situations arise quite naturally in the study of the Blaschke Problem in Differential Geometry; Section II gives some of this background information. Most of the time, in these applications, E is itself a round sphere but occasionally it is not, and in Section V the relatively unexplored area of great k-sphere fibrations of arbitrary manifolds is addressed.

If $E \subseteq S^N$ is fibred by great k-spheres there is a hierarchy of three questions, in increasing order of difficulty, which may be posed:

1) Given two such fibrations, are they topologically equivalent?

2) If they are topologically equivalent, is

it possible to deform one to the other through a one-parameter family of such fibrations?

3) What is the homotopy type of the space of all such fibrations?

In general, even question 1 remains unanswered for all but the simplest cases. Recently all three questions were answered for great circle fibrations of the round 3-sphere [G-W].

Most of this thesis is concerned with a search for answers to these questions.

A good example to keep in mind is that of 3-sphere fibrations of the 7-sphere. There are infintely many topologically inequivalent smooth 3-sphere fibrations of the 7-sphere [MI]. By our Theorem B, each such fibration may be pictured as a fibration by great 3-spheres, provided we choose a suitable embedding of the 7-sphere into a large dimensional sphere S^N. If we insist that the 7-sphere appear as the unit sphere in $\mathbb{R}^8$, then every smooth fibration of it by great 3-spheres is topologically equivalent to the Hopf fibration [G-W-Y]. So while the question of the topological equivalence of all 3-sphere fibrations of the 7-sphere has a negative answer, if one restricts the question to the topological equivalence of all great 3-sphere fibrations of S^7 the answer becomes affirmative.

This illustrates the general expectation, namely, when we lower the dimension of the sphere in which we permit the total space to be embedded or place geometric constraints on the topological type of the total space we correspondingly restrict the bundles whose fibres can thus be made into great k-spheres. It is in this way that the geometric theory departs from the topological theory.

The study of great circle fibrations of round spheres is of particular importance because of its direct connection with the identification of Blaschke manifolds modelled on complex projective spaces (see Section II). In such purposes it is essential to answer Question 1, in this case: are all great circle fibrations of a round sphere topologically equivalent?

If $F : S^1 \to S^{2n-1} \to M_F$ is a great circle fibration of the unit sphere in $\mathbb{R}^{2n}$, we may orient the fibres and then embed the base space M_F in a natural way as a submanifold of the Grassmann manifold $\tilde{G}_2\mathbb{R}^{2n}$ of oriented 2-planes in 2n-space, the embedding denoted by i_F. We show that there are two embeddings of complex projective n-1 space, $\mathbb{C}P^{n-1}$, into $\tilde{G}_2\mathbb{R}^{2n}$, distinct up to homotopy inside $\tilde{G}_2\mathbb{R}^{2n}$, which represent in the above fashion two different versions of the classical Hopf fibration of S^{2n-1} by great circles. Denoting these embeddings by i_{H_1} and i_{H_2}, in Section III we prove

THEOREM A. There exists a homotopy $g : I \times \mathbb{C}P^{n-1} \longrightarrow \tilde{G}_2\mathbb{R}^{2n}$ such that g_0 equals one of i_{H_1} or i_{H_2} and $g_1 = i_F \cdot h_F$, where $h_F : \mathbb{C}P^{n-1} \longrightarrow M_F$ is a homotopy equivalence.

In other words, given a fibration F of S^{2n-1} by oriented great circles, one of the two Hopf fibrations H_1 and H_2 can be selected (depending on F), and its orbit space deformed within the Grassmann manifold until it coincides, via a homotopy equivalence, with the orbit space of F. This is a step towards proving that every great circle fibration of a round sphere is topologically equivalent to a Hopf fibration.

If $E \subseteq S^{2n-1}$ is a fibration by great (n-1)-spheres then the notion of two $(n-1)$-sphere fibres being orthogonal makes sense. In [G-W] it is proven that every great circle fibration of S^3 has an orthogonal pair of fibres. From an analysis of the proof of this result and other theorems in [G-W] one concludes that this fact is equivalent to the Borsuk-Ulam theorem for maps of S^2 to $\mathbb{R}^2$. Based on this it seems natural to ask if every great 3-sphere fibration of S^7, and every great 7-sphere fibration of S^{15}, should likewise have an orthogonal pair of fibres. In Section IV we show that this need not be the case by giving an

EXAMPLE. There exists a great 3-sphere fibration of S^7 with no orthogonal pair of fibres.

A similar approach provides a corresponding example for great 7-sphere fibrations of S^{15}.

In Section V, we study great sphere fibrations of more general manifolds. We begin by proving

THEOREM B. Let $\xi : S^k \to E \to B$ be a smooth k-sphere bundle with group $O(k+1)$ over the compact base space B. Then the total space E can be smoothly embedded into S^N

for N sufficiently large so that each k-sphere fibre becomes a great k-sphere in S^N.

In other words, all reasonable k-sphere bundles can be pictured with great k-sphere fibres by embedding the total space into a large dimensional sphere.

The results in [G-W] and [G-W-Y] deal with fibrations of the round sphere S^n by great k-spheres. The next simplest case to study seems to be $S^p \times S^q$ embedded in S^{p+q+1} by dividing all lengths in the product metric by $\sqrt{2}$. The rest of Section V is devoted to a study of fibrations of $S^p \times S^q$ by great k-spheres.

After a few general pronouncements about such fibrations, we begin sampling the theory for small values of p, q and k. Great circle fibrations of $S^1 \times S^3$ prove to be interesting and some elementary questions about them remain unanswered. Another sample: $S^6 \times S^{13}$ admits no fibrations by great k-spheres for any $k \geq 1$ (while it obviously admits fibrations by 1-spheres, 6-spheres and 13-spheres if we drop the restriction that the fibration be by great spheres in S^{20}).

By far the richest and most satisfying theory we develop is for fibrations of $S^3 \times S^3$ by great 3-spheres. We completely answer the three questions posed at the beginning when we prove

THEOREM C. The space of all oriented great 3-sphere fibrations of $S^3 \times S^3$ deformation retracts to the subspace of "Hopf fibrations" and has the homotopy type of a disjoint union of four copies of real projective 3-space, $\mathbb{RP}^3$.

In the course of proving this we also get:

1) There is a 2 to 1 correspondence between distance decreasing maps from S^3 to S^3 and great 3-sphere fibrations of $S^3 \times S^3$.
2) These fibrations are smooth if and only if the distance decreasing map is smooth and the norm of its differential is strictly less than 1.
3) Every such fibration has an orthogonal pair of fibres.

These are analogous to results obtained in [G-W] for great circle fibrations of S^3.

Using Theorem C we prove

THEOREM D. Every smooth great 3-sphere fibration of $S^3 \times S^3$ can be extended to a smooth great 3-sphere fibration of S^7.

SECTION II

The interest in the topological equivalence of fibrations of S^{n-1} by great subspheres stems from work on the Blaschke conjecture. If M^n is a closed Riemannian manifold of dimension n, and $\alpha(t) = \exp_p(tv)$ is a geodesic with initial direction v, then the cut point of p along α is the last point on α to which the geodesic α minimizes distance from p. The cut locus of p, $C(p)$, is the set of all cut points along any geodesic emanating from p. M^n is called a Blaschke manifold if the distance from p to $C(p)$ is a constant and this constant is independent of p. It is known that the cut locus of any point in an arbitrary Blaschke manifold M^n is either a point or a smooth submanifold of dimension $n-1$, $n-2$, or $n-4$, or else if $n = 16$, the cut locus can be 8-dimensional. Such an M^n is said to be modelled on S^n, $\mathbb{R}P^n$, $\mathbb{C}P^{n/2}$, $\mathbb{H}P^{n/4}$ = quaternionic projective space of real dimension n, or $\mathbb{C}aP^2$ = Caley projective plane if $n = 16$, respectively. It has been a long standing conjecture, first attributed to Blaschke in 1921 for the case $n = 2$ [BL], that any Blaschke manifold is isometric (up to change of scale) to its model space. The conjecture has been answered in the affirmative for M^n modelled on S^n or $\mathbb{R}P^n$ and is open in all remaining cases [GR, BE, KA, WE, YA-1].

The Blaschke conjecture is purely geometric in nature and nothing is presumed about the topology of M^n. Certainly a necessary condition for the truth of the conjecture is that M^n be homeomorphic to its model space.

Let M^n be a Blaschke manifold, $p \in M$, and M_p the tangent space to M at p. The exponential map $\exp_p\colon M_p \longrightarrow M$ takes a round ball $B(p)$ centered at $0 \in M_p$ onto M and takes $\partial B(p)$ to the cut locus $C(p)$. The following theorem which describes the exponential mapping for a Blaschke manifold allows one to begin trying to characterize M^n topologically. It is due to Omori in the real analytic case [OM], and to Nakagawa and Shiohama in the differentiable case [N-S 1, N-S 2].

THEOREM. If M^n is a Blaschke manifold, then the cut locus $C(p)$ to any point $p \in M$ is a smooth submanifold of M and $\exp_p\colon \partial B(p) \longrightarrow C(p)$ is a smooth fibre bundle. Moreover, the fibres are great subspheres of the round sphere $\partial B(p)$.

So M^n is homeomorphic to the "mapping cone" of the fibration $\exp_p$: $\partial B(p) \longrightarrow C(p)$, and to prove that any Blaschke manifold modelled on $\mathbb{C}P^n$, $\mathbb{H}P^n$, or $\mathbb{C}aP^2$ is homeomorphic to its model space, it would be sufficient to prove the

CONJECTURE. Any smooth fibration of S^{n-1} by great subspheres is topologically equivalent to a Hopf fibration.

This conjecture has recently been shown to have a positive answer for $n < 9$ and also for $n = 16$ and fibrations by great 7-spheres [G-W-Y].

The key idea in [G-W-Y], first suggested by Warner, is to capitalize on the fact that the fibres are great subspheres by viewing the orbit space inside the Grassmann manifold and studying its position there. Sections III and V of this thesis pursue this point of view.

SECTION III

Let $F : S^1 \to S^{2n-1} \to M_F$ be an arbitrary smooth great circle fibration of S^{2n-1}. We lose no generality in assuming the fibres are oriented and thus the base space has a natural embedding into $\tilde{G}_2\mathbb{R}^{2n}$, $i_F : M_F \to \tilde{G}_2\mathbb{R}^{2n}$. As the base space of a great circle fibration of S^{2n-1}, M_F has the homotopy type of $\mathbb{C}P^{n-1}$ ([G-W-Y), Sect 10) so let $h_F: \mathbb{C}P^{n-1} \to M_F$ be a homotopy equivalence. In this section we show that there are two embeddings, i_{H_1} and i_{H_2}, of $\mathbb{C}P^{n-1}$ into $\tilde{G}_2\mathbb{R}^{2n}$ that correspond to Hopf fibrations of S^{2n-1} and are distinct up to homotopy in $\tilde{G}_2\mathbb{R}^{2n}$. Our main result is:

THEOREM A. There exists a homotopy $g : I \times \mathbb{C}P^{n-1} \to \tilde{G}_2\mathbb{R}^{2n}$ such that g_0 equals one of i_{H_1} or i_{H_2} and $g_1 = i_F \cdot h_F$.

The proof proceeds by first showing $i_{F_*}(H_*(M_F)) = i_{H_{j*}}(H_*(\mathbb{C}P^{n-1}))$ for j either 1 or 2 (all homology and cohomology groups in this section have coefficients in $\mathbb{Z}$). Using this information, the homotopy information of $\tilde{G}_2\mathbb{R}^{2n}$ and the Hurewicz homomorphism from homotopy to homology we conclude that the obstructions to constructing the homotopy g all vanish.

NOTATION AND CONVENTIONS.

1) $e_1, e_2, \ldots, e_{2n}$ denotes the standard orthonormal basis vectors for $\mathbb{R}^{2n}$. We suppose $\mathbb{R}^{2n} \simeq \mathbb{C}^n$ with $ie_{2j-1} = e_{2j}$ and $ie_{2j} = -e_{2j-1}$, $1 \leq j \leq n$, $i = \sqrt{-1}$.

2) H_1: $S^1 \longrightarrow S^{2n-1} \xrightarrow{\pi} \mathbb{C}P^{n-1}$ is the standard Hopf fibration. $\pi(z_1, z_2, \ldots, z_n) = \pi(w_1, w_2, \ldots, w_n)$ if and only if there exists $\lambda \in S^1 \subseteq \mathbb{C}$ with $z_i = \lambda w_i$, $1 \leq i \leq n$. This bundle is oriented by the usual counterclockwise orientation on $\mathbb{C}$. An equivalence class in S^{2n-1} under the projection π is denoted by square brackets, $[z_1, z_2, \ldots, z_n] =$ $= \pi^{-1} \cdot \pi(z_1, z_2, \ldots, z_n)$.

3) $A : \tilde{G}_2\mathbb{R}^{2n} \longrightarrow \tilde{G}_2\mathbb{R}^{2n}$ will be the involution which sends an oriented 2-plane to the same physical plane but with the opposite orientation.

4) $P \in \tilde{G}_2\mathbb{R}^{2n}$ will always denote the oriented 2-plane spanned by the ordered frame (e_1, e_2). $A(P) = P_{op}$ is spanned by the ordered frame (e_2, e_1).

5) Let $U_P = \{Q \in \tilde{G}_2\mathbb{R}^{2n} : Q \cap P^{\perp} = \{0\}\}$. For local coordinates about P we take the map

$$\mu : U_P \longrightarrow \mathbb{R}^{2(2n-2)} = \mathrm{Hom}(P, P^{\perp})$$

where $\mu(Q)$ is a $(2n-2) \times 2$ matrix such that the graph of $\mu(Q)$ as a map from P with basis $\{e_1, e_2\}$ to $P^{\perp}$ with basis $\{e_3, e_4, \ldots, e_{2n}\}$ is the plane Q. Here we assume

$$(x_1, x_2, \ldots, x_{2n-2}, x_{2n-1}, \ldots, x_{2(2n-2)}) \longrightarrow \begin{bmatrix} x_1 & x_{2n-1} \\ x_2 & x_{2n} \\ \vdots & \vdots \\ x_{2n-2} & x_{2(2n-2)} \end{bmatrix}$$

identifies $\mathbb{R}^{2(2n-2)}$ with $\mathrm{Hom}(P, P^{\perp})$.

6) Finally we assume $\tilde{G}_2\mathbb{R}^{2n}$ is oriented so that $\mu : U_P \longrightarrow \mathbb{R}^{2(2n-2)}$ is orientation preserving, where $\mathbb{R}^{2(2n-2)}$ has its usual orientation (we will see below that $\tilde{G}_2\mathbb{R}^{2n}$ is an orientable manifold).

We begin by assembling some results on the topology of $\tilde{G}_2\mathbb{R}^{2n}$. $\tilde{G}_2\mathbb{R}^{2n}$ is the base space of an oriented circle fibration, the Stiefel bundle,

$$B : S^1 \longrightarrow V_2\mathbb{R}^{2n} \xrightarrow{p} G_2\mathbb{R}^{2n}$$

where $V_2\mathbb{R}^{2n}$ is the Stiefel manifold of oriented, orthonormal 2-frames in $\mathbb{R}^{2n}$, $p((v_1,v_2))$ is the oriented 2-plane spanned by the ordered frame (v_1,v_2). From ([ST], Sect 25.6) we get

$$\pi_i(V_2\mathbb{R}^{2n}) = 0 \quad 1 \leq i \leq 2n-3$$

$$\pi_{2n-2}(V_2\mathbb{R}^{2n}) \simeq \mathbb{Z} .$$

The exact homotopy sequence of the Stiefel bundle now implies $\pi_1(V_2\mathbb{R}^{2n}) = \pi_1(G_2\mathbb{R}^{2n}) = 0$ so both manifolds are orientable.

By the Hurewicz Isomorphism Theorem we conclude

$$H_i(V_2\mathbb{R}^{2n}) = 0 \quad 1 \leq i \leq 2n-3$$

$$H_{2n-2}(V_2\mathbb{R}^{2n}) \simeq \mathbb{Z} .$$

Now $\dim V_2\mathbb{R}^{2n} = \dim G_2\mathbb{R}^{2n} + 1 = 2(2n-2) + 1$ so since all $H_i(V_2\mathbb{R}^{2n})$, $i \leq 2n-2$ are free, by Poincare duality we get:

$$H_i(V_2\mathbb{R}^{2n}) \simeq H^i(V_2\mathbb{R}^{2n}) \simeq \begin{cases} \mathbb{Z} & i = 0\ ,\ 2n-2\ ,\ 2n-1\ ,2(2n-2)+1 \\ 0 & \text{otherwise} \end{cases}$$

LEMMA 3.1.

$$H_{2j}(\tilde{G}_2\mathbb{R}^{2n}) \simeq H^{2j}(\tilde{G}_2\mathbb{R}^{2n}) \simeq \begin{cases} \mathbb{Z} & 0 \leq j \leq 2n-2\ ,\ j \neq n-1 \\ \mathbb{Z} + \mathbb{Z} & j = n-1 \end{cases}$$

and 0 in all other dimensions. Furthermore, one generator for $H^{2j}(\tilde{G}_2\mathbb{R}^{2n})$, $1 \leq j \leq n-1$ in $e(B)^j$ where $e(B)$ is the Euler class of the Stiefel bundle.

PROOF: From the Gysin sequence of the Stiefel bundle, for $1 \leq i \leq 2n-4$, we get

$$0 = H^i(V_2\mathbb{R}^{2n}) \longrightarrow H^{i-1}(\tilde{G}_2\mathbb{R}^{2n})$$

$$\xrightarrow{\cup e(B)} H^{i+1}(\tilde{G}_2\mathbb{R}^{2n}) \longrightarrow H^{i+1}(V_2\mathbb{R}^{2n}) = 0$$

Hence

$$H^i(\tilde{G}_2\mathbb{R}^{2n}) = \begin{cases} 0 & i \ \text{odd},\ i \leq 2n-3 \\ \mathbb{Z} & i = 2j\ ,\ i \leq 2n-4 \text{ with } e(B)^j \text{ as generator.} \end{cases}$$

For $i = 2n-3$:

$$
\begin{array}{ccccc}
H^{2n-3}(V_2\mathbb{R}^{2n}) & \longrightarrow & H^{2n-4}(\tilde{G}_2\mathbb{R}^{2n}) & \xrightarrow{\cup e(B)} & H^{2n-2}(\tilde{G}_2\mathbb{R}^{2n}) \\
\| & & \| & & \\
0 & & \mathbb{Z} & &
\end{array}
$$

$$
\begin{array}{ccccc}
\longrightarrow & H^{2n-2}(V_2\mathbb{R}^{2n}) & \longrightarrow & H^{2n-3}(\tilde{G}_2\mathbb{R}^{2n}) \\
& \| & & \| \\
& \mathbb{Z} & & 0
\end{array}
$$

By exactness we conclude $H^{2n-2}(\tilde{G}_2\mathbb{R}^{2n}) \simeq \mathbb{Z} + \mathbb{Z}$ with $e(B)^{n-1}$ as one generator. Now apply duality, using the fact that all groups are torsion-free, to conclude the proof. QED.

Our next objective is to find specific representatives of generators for $H_{2n-2}(\tilde{G}_2\mathbb{R}^{2n}) \simeq \mathbb{Z} + \mathbb{Z}$. Let S^{2n-2} denote the unit (2n-2)-sphere contained in $S^{2n-1} \subseteq \mathbb{R}^{2n}$ determined by $S^{2n-1} \cap e_1{}^{\perp}$. There is a map $f : S^{2n-2} \longrightarrow V_2\mathbb{R}^{2n}$ given by $f(x) = (e_1, x)$ and f represents a generator of $\pi_{2n-2}(V_2\mathbb{R}^{2n}) \simeq H_{2n-2}(V_2\mathbb{R}^{2n})$ ([ST], Sect 25.6).

LEMMA 3.2. If $[S^{2n-2}]$ and $[\mathbb{C}P^{n-1}]$ denote fundamental cycles in $H_{2n-2}(S^{2n-2})$ and $H_{2n-2}(\mathbb{C}P^{n-1})$ respectively, then $(p\cdot f)_*([S^{2n-2}])$ and $i_{H_1 *}([\mathbb{C}P^{n-1}])$ generate $H_{2n-2}(\tilde{G}_2\mathbb{R}^{2n})$.

PROOF: From the dual Gysin sequence in homology for the Stiefel bundle we have

$$0 = H_{2n-3}(\tilde{G}_2\mathbb{R}^{2n}) \longrightarrow H_{2n-2}(V_2\mathbb{R}^{2n}) \xrightarrow{p_*} H_{2n-2}(\tilde{G}_2\mathbb{R}^{2n})$$

so since $f_*([S^{2n-2}])$ represents a generator of $H_{2n-2}(V_2\mathbb{R}^{2n})$ it follows by injectivity of the map p_* that $(p\cdot f)_*([S^{2n-2}])$ represents a generator of $H_{2n-2}(G_2\mathbb{R}^{2n})$.

So suppose $\alpha = (p\cdot f)_*([S^{2n-2}])$ and β represent generators of $H_{2n-2}(\tilde{G}_2\mathbb{R}^{2n})$, and $i_{H_{1*}}([\mathbb{C}P^{n-1}]) = a\alpha + b\beta$. If we can show $b = \pm 1$ we are done for then we conclude that the classes of $(p\cdot f)_*([S^{2n-2}])$ and $i_{H_{1*}}([\mathbb{C}P^{n-1}])$ also generate $H_{2n-2}(\tilde{G}_2\mathbb{R}^{2n})$.

Recalling from Section I that all three of the questions about great sphere fibrations have been completely answered for great circle fibrations of S^3 we gain nothing by including the case $n = 2$ and assume henceforth that $n > 2$. This implies $2n-2 > 2$ and $(p\cdot f)^*(e(B)) \in H^2(S^{2n-2}) = 0$ hence

$$\langle\alpha, e(B)^{n-1}\rangle = \langle[S^{2n-2}], (p\cdot f)^*(e(B))^{n-1}\rangle = 0$$

where $<\,,\,>$ denote the evaluation of a cochain on a chain. Therefore since $e(B)^{n-1}$ represents a generator of $H^{2n-2}(\tilde{G}_2\mathbb{R}^{2n})$ we must have $\langle c_1\alpha+c_2\beta, e(B)^{n-1}\rangle = c_2$.

So it remains to show that $\langle i_{H_{1*}}([\mathbb{C}P^{n-1}]), e(B)^{n-1}\rangle = \pm 1$. To get this we note that the map $i_{H_1}\colon \mathbb{C}P^{n-1} \longrightarrow \tilde{G}_2\mathbb{R}^{2n}$ is covered by a map of total spaces, $\tilde{i}_{H_1}\colon S^{2n-1} \longrightarrow V_2\mathbb{R}^{2n}$ such that the pair $(\tilde{i}_{H_1}, i_{H_1})\colon H_1 \longrightarrow B$ is a bundle map. Hence $i_{H_1}{}^*(e(B)) = e(H_1)$ is the Euler class of the Hopf bundle. From the Gysin sequence of the Hopf bundle, $e(H_1)$ generates the cohomology algebra $H^*(\mathbb{C}P^{n-1}) \approx \mathbb{Z}[e(H_1)]/(e(H_1)^n]$, and hence, for appropriate choice of orientation we may assume $1 = \langle[\mathbb{C}P^{n-1}], e(H_1)^{n-1}\rangle = \langle i_{H_*}([\mathbb{C}P^{n-1}]), e(B)^{n-1}\rangle$. QED.

Our plan to compute the homology class of $i_{F_*}([M_F]) \in H_{2n-2}(\tilde{G}_2\mathbb{R}^{2n})$ with respect to the basis of Lemma 3.2 is to compute intersection products. Since $\dim \tilde{G}_2\mathbb{R}^{2n} = 2(2n-2) \equiv 0 \pmod 4$ there is a symmetric, bilinear form

$$q_n\colon H_{2n-2}(\tilde{G}_2\mathbb{R}^{2n}) \times H_{2n-2}(\tilde{G}_2\mathbb{R}^{2n}) \longrightarrow \mathbb{Z}\,.$$

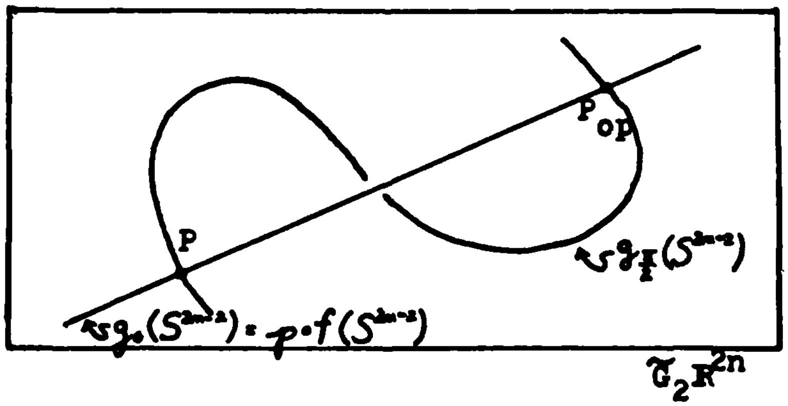
Poincare dual to cup products in cohomology, gotten by computing intersection products of representative cycles ([DO], Chap 8, Sect 13). In the next series of lemmas we compute the matrix for q_n with respect to our basis for $H_{2n-2}(\tilde{G}_2\mathbb{R}^{2n})$.

LEMMA 3.3. $p\cdot f(S^{2n-2})$ is a smoothly embedded submanifold and it has self-intersection number +2 in $\tilde{G}_2\mathbb{R}^{2n}$.

PROOF: Recall that $p : V_2\mathbb{R}^{2n} \longrightarrow \tilde{G}_2\mathbb{R}^{2n}$, and $P = \mathrm{span}(e_1, e_2)$.

That $p\cdot f$ is 1-1 onto its image is clear, that it is a smooth embedding will be evident from the computation of the self-intersection number.

Our goal is to deform the image of S^{2n-2} off of itself to a new position and show that the two images intersect each other transversally in precisely two points with the same orientation at each.



Define $g : [0,\pi] \times S^{2n-2} \longrightarrow \tilde{G}_2\mathbb{R}^{2n}$

$g_\theta(x_1,\ldots,x_{2n-1})$

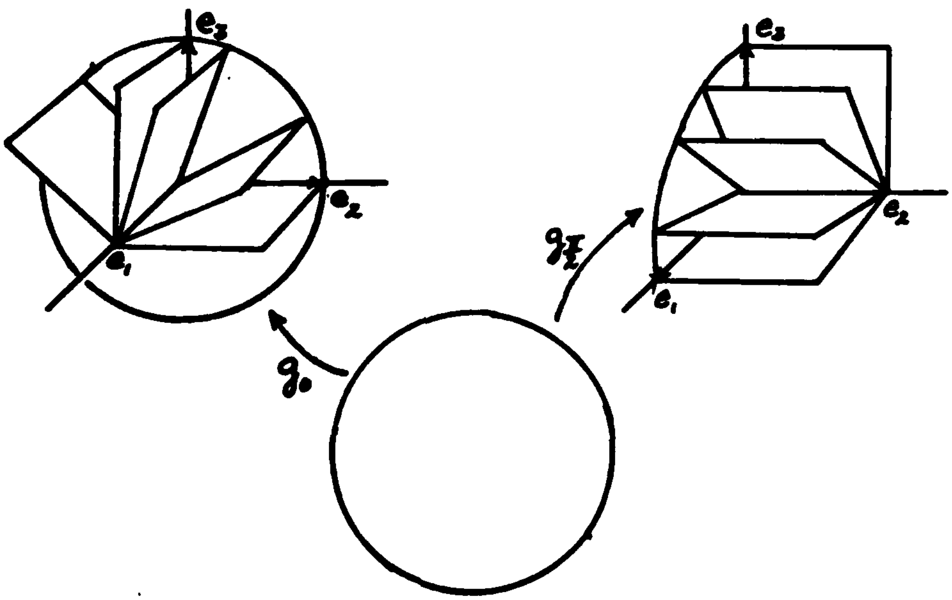
$= p((\cos\theta,-\sin\theta,0,\ldots,0),(x_1\sin\theta,x_1\cos\theta,x_2,\ldots,x_{2n-1}))$.

We have

$$g_0(1,0,\ldots,0) = g_{\pi/2}(1,0,\ldots,0) = P$$

$$g_0(-1,0,\ldots,0) = g_{\pi/2}(-1,0,\ldots,0) = P_{op}$$

and since $g_0(S^{2n-2})$ = all 2-planes in $\mathbb{R}^{2n}$ containing the vector e_1, and $g_{\pi/2}(S^{2n-2})$ = all 2-planes in $\mathbb{R}^{2n}$ containing the vector e_2, P and P_{op} are the only possible points where $g_0(S^{2n-2})$ intersects $g_{\pi/2}(S^{2n-2})$.



Since g_0 and $g_{\pi/2}$ are both injective there are exactly two points $x_p = (1,0,\ldots,0)$ and $y_p = (-1,0,\ldots,0)$ in S^{2n-2} with $g_0(x_p) = g_{\pi/2}(x_p) = P$ and $g_0(y_p) = g_{\pi/2}(y_p) = P_{op}$.

Now we show that the intersections at these two points are transverse and that the direct sum orientations at both points are consistent with our original choice of orientation on $\tilde{G}_2\mathbb{R}^{2n}$.

We digress to examine the coordinate chart $\varphi : U_p \longrightarrow \mathbb{R}^{2(2n-2)}$.

$$\varphi^{-1}\left(\begin{bmatrix} a_{1,1} & a_{1,2} \\ \vdots & \vdots \\ a_{2n-1,1} & a_{2n-1,2} \end{bmatrix}\right)$$

is the 2-plane Q which is the graph of this matrix viewed as a map from P with basis (e_1,e_2) to $P^{\perp}$ with basis $(e_3,e_4,\ldots,e_{2n})$. So we see that

$$Q = \text{span}((1,0,a_{1,1},a_{2,1},\ldots,a_{2n-1,1}),(0,1,a_{1,2},a_{2,2},\ldots,a_{2n-1,2})). \quad (*)$$

Now we take $\tau_1 : \{(x_1,\ldots,x_{2n-1}) \in S^{2n-2} : x_1 > \frac{1}{2}\} \longrightarrow \mathbb{R}^{2n-2}$,

$$\tau_1(x_1,\ldots,x_{2n-1}) = \left(\frac{x_2}{x_1},\frac{x_3}{x_1},\ldots,\frac{x_{2n-1}}{x_1}\right)$$

as a coordinate chart in a neighborhood of $x_p = (1,0,\ldots,0) \in S^{2n-2}$. Since

$$\begin{aligned} g_0(x_1,\ldots,x_{2n-1}) &= p((1,0,\ldots,0),(0,x_1,x_2,\ldots,x_{2n-1})) \\ &= \mathrm{span}\left((1,0,\ldots,0),\left(0,1,\frac{x_2}{x_1},\ldots,\frac{x_{2n-1}}{x_1}\right)\right) \end{aligned}$$

if $x_1 \neq 0$ from (*) we conclude

$$\varphi\cdot g_0\cdot \tau_1^{-1}(y_1,y_2,\ldots,y_{2n-2}) = \begin{bmatrix} 0 & y_1 \\ 0 & y_2 \\ \vdots & \vdots \\ 0 & y_{2n-2} \end{bmatrix}$$

likewise we get

$$\varphi\cdot g_{\frac{\pi}{2}}\cdot \tau_1^{-1}(y_1,\ldots,y_{2n-2}) = \begin{bmatrix} y_1 & 0 \\ y_2 & 0 \\ \vdots & \vdots \\ y_{2n-2} & 0 \end{bmatrix}$$

The map φ works perfectly well in a neighborhood of P_{op} and we have τ_{-1} defined exactly as τ_1 on the corresponding neighborhood of $y_p = (-1,0,\ldots,0) \in S^{2n-2}$ hence we get the identical result computing in a neighborhood of y_p . We conclude immediately that the intersections at P and P_{op} are transverse, since if $T_0\mathbb{R}^{2n-2}$ denotes the tangent space to $\mathbb{R}^{2n-2}$ at the origin we have

$$d(\varphi \cdot g_{\frac{\pi}{2}} \cdot \tau_1^{-1})_0(T_0\mathbb{R}^{2n-2}) \oplus d(\varphi \cdot g_0 \cdot \tau_1^{-1})_0(T_0\mathbb{R}^{2n-2}) = T_0\mathbb{R}^{2(2n-2)} .$$

Its also clear that whatever orientation is chosen for $\mathbb{R}^{2n-2}$ the direct sum of the two induced orientations in $\mathbb{R}^{2(2n-2)}$ will be consistent with the standard orientation on $\mathbb{R}^{2(2n-2)}$. So if we fix an orientation of S^{2n-2} , since $2n-2$ is even, the orientations induced on $\mathbb{R}^{2n-2}$ by τ_1 and τ_{-1} will be opposite but this fact allows us to conclude that the direct sum orientations in $\mathbb{R}^{2(2n-2)}$ computed using τ_1 or τ_{-1} will agree.

So based on our choice of orientation for $\mathcal{G}_2\mathbb{R}^{2n}$, we can say that the intersection number at P is $+1$. It remains to show that the orientation on $U_{P_{op}}$ such that $\varphi : U_{P_{op}} \longrightarrow \mathbb{R}^{2(2n-2)}$ is orientation preserving is consistent with the global orientation of $\mathcal{G}_2\mathbb{R}^{2n}$, for then the intersection number at P_{op} will also be +1 .

SUBLEMMA 3.4. $G_k\mathbb{R}^m$, $k < m$, is orientable if and only if m is even.

PROOF: Let $Q = \text{span}(e_1,e_2,\ldots,e_k) \in G_k\mathbb{R}^m$ and define

$$\rho : [0,\pi]\times \mathbb{R}^m \longrightarrow \mathbb{R}^m \text{ with } \rho_\theta \in SO(m) \text{ for all } \theta$$

$$\rho_\theta(e_1) = \cos\theta e_1 - \sin\theta e_m$$

$$\rho_\theta(e_m) = \sin\theta e_1 + \cos\theta e_m$$

$$\rho_\theta(e_i) = e_i\ ,\ 1 < i < m$$

ρ_θ induces a map $\tilde{G}_k\mathbb{R}^m \longrightarrow \tilde{G}_k\mathbb{R}^m$ which we also denote by ρ_θ. Note that $\rho_\pi(Q) = Q_{op} = A(Q)$.

Let $\tilde{\Phi} : U_Q \longrightarrow \mathbb{R}^{k(m-k)} \simeq \text{Hom}(Q,Q^\perp)$ be the usual coordinate chart centered at Q, where $U_Q = \{R : R\cap Q^\perp = \{0\}\}$. Since ρ_π sends e_1 to $-e_1$ and e_m to $-e_m$, viewing a k-plane $R \in U_Q$ as the graph of a linear transformation from the basis $(e_1,\ldots,e_k)$ to the basis $(e_{k+1},\ldots,e_m)$, ρ_π changes the sign on a vector in the domain (e_1) and a vector in the range (e_m). Therefore we conclude

$$\tilde{\varphi}\cdot A\cdot \rho_\pi \cdot \tilde{\varphi}^{-1}\left(\begin{bmatrix} a_{11} & \cdots & a_{1k} \\ \vdots & & \vdots \\ a_{n-k,1} & \cdots & a_{n-k,k} \end{bmatrix}\right) =$$

$$\begin{bmatrix} -a_{11} & a_{12} & \cdots & a_{1k} \\ -a_{21} & \vdots & & \\ \vdots & & & \\ -a_{n-k-1,1} & a_{n-k-1,2} & \cdots & a_{n-k-1,k} \\ a_{n-k,1} & -a_{n-k,2} & \cdots & -a_{n-k,k} \end{bmatrix}$$

So the total number of minus signs introduced is $(m-k-1) + (k-1) = m - 2$ which is even if and only if m is even. ρ_π is homotopic to the identity so it is always orientation preserving, therefore A is orientation preserving if and only if m is even. But $G_k\mathbb{R}^m$ is just the quotient space of $\tilde{G}_k\mathbb{R}^m$ under the action of A, hence $G_k\mathbb{R}^m$ is orientable if and only if A is orientation preserving. QED SUBLEMMA

As we see in the last paragraph of the proof of the sublemma, $G_2\mathbb{R}^{2n}$ orientable implies the two orientations on neighborhoods of P and P_{op} which are each consistent with a fixed orientation on $\mathbb{R}^{2(2n-2)}$ under the map φ, are consistent with a global orientation of $\widetilde{G}_2\mathbb{R}^{2n}$. This completes the proof of LEMMA 3.3.

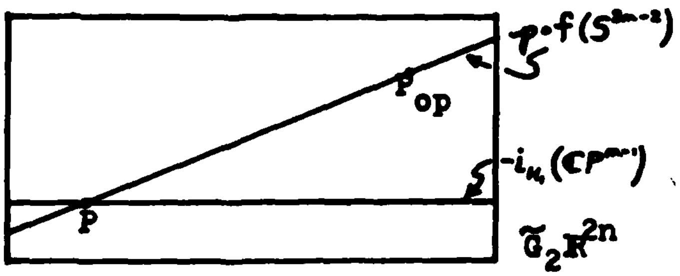
DEFINITION 3.5. For $Q \in \widetilde{G}_2\mathbb{R}^{2n}$, the <u>Bad cone of Q</u> $= \{R \in \widetilde{G}_2\mathbb{R}^{2n}: \ v \in R \cap Q \text{ where } 0 \neq v \in \mathbb{R}^{2n}\}$ = the set of all oriented 2-planes in $\mathbb{R}^{2n}$ which meet Q in at least a line.

REMARK 3.6. Note that $p \cdot f(S^{2n-2}) \subseteq$ Bad cone of P .

LEMMA 3.7. The map $i_F : M_F \longrightarrow G_2\mathbb{R}^{2n}$ is a smooth embedding and $i_F(M_F)$ meets the bad cone of Q , B_Q , transversally, for each $Q \in i_F(M_F)$.

PROOF: ([G-W-Y], Theorem 4.1)

In particular, if F is the Hopf fibration H_1 then with the above remark we conclude $p \cdot f(S^{2n-2}) \pitchfork i_{H_1}(\mathbb{C}P^{n-1})$. Therefore if $\cdot$ denotes the intersection product, we have $(p \cdot f)_*([S^{2n-2}]) \cdot i_{H_{1*}}([\mathbb{C}P^{n-1}]) = \pm 1$.



Now define $[\mathbb{C}P^{n-1}]$ so that $\langle[\mathbb{C}P^{n-1}], e(H_1)^{n-1}\rangle = +1$ and $[S^{2n-2}]$ so that $(p\cdot f)_*([S^{2n-2}])\cdot i_{H_{1_*}}([\mathbb{C}P^{n-1}]) = +1$. In all that follows we assume $(a,b) \in \mathbb{Z} + \mathbb{Z}$ denotes that element of $H_{2n-2}(\tilde{G}_2\mathbb{R}^{2n})$ with representative $a(p\cdot f)_*([S^{2n-2}]) + b i_{H_{1_*}}([\mathbb{C}P^{n-1}])$.

It remains to compute $i_{H_{1_*}}([\mathbb{C}P^{n-1}])\cdot i_{H_{1_*}}([\mathbb{C}P^{n-1}])$ to complete the matrix representation of q_n .

LEMMA 3.8. $i_{H_1}(\mathbb{C}P^{n-1})$ has self-intersection number 0 if n is even, 1 if n is odd ($i_{H_{1_*}}([\mathbb{C}P^{n-1}])\cdot i_{H_{1_*}}([\mathbb{C}P^{n-1}]) \equiv n \bmod 2$).

PROOF: When $n = 2$, $\tilde{G}_2\mathbb{R}^4 \approx S^2 \times S^2$ and $i_{H_1}(\mathbb{C}P^1)$ appears as $S^2 \times \{pt\}$ so $i_{H_1}(\mathbb{C}P^1)$ clearly has self-intersection number 0 [G-W]. This deformation process can be mimicked for $\tilde{G}_2\mathbb{R}^{4k}$ to conclude $\mathbb{C}P^{2k-1}$ can be deformed completely off of itself.

Let

$$T(\theta) = \begin{bmatrix} \cos\theta & 0 & -\sin\theta & 0 \\ 0 & \cos\theta & 0 & \sin\theta \\ \sin\theta & 0 & \cos\theta & 0 \\ 0 & -\sin\theta & 0 & \cos\theta \end{bmatrix} \in SO(4)$$

1) Assume $n = 2m$. Let $S(\theta)$ be the $2n \times 2n$ matrix

$$S(\theta) = \begin{bmatrix} T(\theta) & & & 0 \\ & T(\theta) & & \\ & & \ddots & \\ 0 & & & T(\theta) \end{bmatrix} \quad \text{(m copies of } T(\theta) \text{ on the diagonal).}$$

We apply $S(\theta)$ to the complex lines in $\mathbb{R}^{2n}$ then since $S(\pi/2)$ is homotopic to $S(0) = I_{2n}$ we view it as giving a smooth deformation of $i_{H_1}(\mathbb{C}P^{n-1})$ inside $\tilde{G}_2\mathbb{R}^{2n}$.

The claim is that $S(\pi/2)$ takes complex lines to complex lines, however, with the opposite orientation. Let P_x be the complex line containing the vector $x = (x_1+iy_1, x_2+iy_2, \ldots, x_n+iy_n)$. Clearly we need only examine the effect of $T(\pi/2)$ on $\tilde{x} = (x_1, y_1, x_2, y_2)$.

$$T(\pi/2)\begin{bmatrix} x_1 \\ y_1 \\ x_2 \\ y_2 \end{bmatrix} = \begin{bmatrix} -x_2 \\ y_2 \\ x_1 \\ -y_1 \end{bmatrix}$$

$$T(\pi/2)(i\tilde{x}) = T(\pi/2)\begin{bmatrix} -y_1 \\ x_1 \\ -y_2 \\ x_2 \end{bmatrix} = \begin{bmatrix} y_2 \\ x_2 \\ -y_1 \\ x_1 \end{bmatrix} = -i\begin{bmatrix} -x_2 \\ y_2 \\ x_1 \\ -y_1 \end{bmatrix} = -iT(\pi/2)(\tilde{x})$$

Hence

$$T(\pi/2)\left((a+bi)\begin{bmatrix}x_1\\y_1\\x_2\\y_2\end{bmatrix}\right) = (a-bi)T(\pi/2)\begin{bmatrix}x_1\\y_1\\x_2\\y_2\end{bmatrix} = (a-bi)\begin{bmatrix}-x_2\\y_2\\x_1\\-y_1\end{bmatrix} \quad (3.8.1)$$

So $S(\pi/2)(P_x) = A(P_{x'})$ where $x' = (-x_2, y_2, x_1, -y_1, \ldots, -x_n, y_n, x_{n-1}, -y_{n-1})$ and $S(\pi/2)(i_{H_1}(\mathbb{C}P^{n-1})) = A\cdot i_{H_1}(\mathbb{C}P^{n-1})$. Since $i_{H_1}(\mathbb{C}P^{n-1}) \cap A\cdot i_{H_1}(\mathbb{C}P^{n-1}) = \emptyset$, the case n even is complete.

Before we treat the case n odd we pause to define an alternate complex structure on $\mathbb{R}^{2n}$. If J denotes our original complex structure on $\mathbb{R}^{2n}$ given in the NOTATION AND CONVENTIONS (1), then our new structure, denoted J_P, may loosely be described as $J_P = J|_P - J|_{P^\perp}$. Rigorously, we define

$$\left.\begin{aligned} J_P(e_1) &= J(e_1) = e_2 \\ J_P(e_2) &= J(e_2) = -e_1 \end{aligned}\right\}\text{usual complex structure on } P$$

$$\left.\begin{aligned} J_P(e_{2j-1}) &= -J(e_{2j-1}) = -e_{2j} \\ J_P(e_{2j}) &= -J(e_{2j}) = e_{2j-1} \end{aligned}\right\}\begin{array}{l}\text{for } 2 \le j \le n, \\ \text{"opposite" complex} \\ \text{structure on } P^\perp\end{array}$$

Now let $G : S^1 \longrightarrow S^{2n-1} \xrightarrow{\pi_G} M_G$ denote the great circle fibration of S^{2n-1} determined by the complex structure J_p. $\pi_G(z_1,z_2,\ldots,z_n) = \pi_G(w_1,w_2,\ldots,w_n)$ if and only if there exists $\lambda \in S^1 \subseteq \mathbb{C}$ with $z_1 = \lambda w_1$ and $z_i = \bar{\lambda} w_i$, $2 \leq i \leq n$. The equivalence classes in S^{2n-1} under the projection π_G are denoted by curly brackets, $\{z_1:z_2:\ldots:z_n\} = \pi_G^{-1}\cdot\pi_G(z_1,z_2,\ldots,z_n)$. Clearly M_G is diffeomorphic to $\mathbb{C}P^{n-1}$ (if $r : \tilde{G}_2\mathbb{R}^{2n} \longrightarrow \tilde{G}_2\mathbb{R}^{2n}$ is the map that sends a 2-plane Q to the reflection of Q in the e_1-axis then its easy to see that $A\cdot i_{H_1}(\mathbb{C}P^{n-1}) = r\cdot i_G(M_G))$.

CLAIM. $i_G(M_G) \cap i_{H_1}(\mathbb{C}P^{n-1}) = \{P\}$ and the intersection is transverse.

PROOF: Let $\psi_H\colon V_p \longrightarrow \mathbb{R}^{2n-2}$ be the usual coordinate chart about $P = [1:0:\ldots:0] \in \mathbb{C}P^{n-1}$, $\psi_H([z_1:z_2:\ldots:z_n]) = (z_2/z_1, z_3/z_1,\ldots,z_n/z_1) \in \mathbb{C}^{n-1}$. Let $\psi_G\colon W_p \longrightarrow \mathbb{R}^{2n-2}$ be the usual coordinate chart about $P = \{1:0:\ldots:0\} \in M_G$, $\psi_G(\{z_1:z_2:\ldots:z_n\}) = (z_2/\bar{z}_1, z_3/\bar{z}_1,\ldots,z_n/\bar{z}_1) \in \mathbb{C}^{n-1}$. We have

$$\varphi\cdot i_{H_1}\cdot\psi_H^{-1}(x_1,y_1,\ldots,x_{n-1},y_{n-1}) = \begin{bmatrix} x_1 & -y_1 \\ y_1 & x_1 \\ \vdots & \vdots \\ x_{n-1} & -y_{n-1} \\ y_{n-1} & x_{n-1} \end{bmatrix} \quad (3.8.2)$$

$$\varphi\cdot i_G\cdot\psi_G^{-1}(x_1,y_1,\ldots,x_{n-1},y_{n-1}) = \begin{bmatrix} x_1 & y_1 \\ y_1 & -x_1 \\ \vdots & \vdots \\ x_{n-1} & y_{n-1} \\ y_{n-1} & -x_{n-1} \end{bmatrix}$$

Its easy now to verify that the intersection at P is transverse and that in U_p , P is the only point where $i_{H_1}(\mathbb{C}P^{n-1})$ intersects $i_G(M_G)$. If $R \in i_{H_1}(\mathbb{C}P^{n-1}) \cap \cap(\tilde{G}_2\mathbb{R}^{2n}-U_p)$ then $R \subseteq P^{\perp}$ so clearly $R \not\subset i_G(M_G)$.

We comment that in fact $i_G(M_G)$ and $i_{H_1}(\mathbb{C}P^{n-1})$ are orthogonal totally geodesic submanifolds of $\tilde{G}_2\mathbb{R}^{2n}$.

2) Now suppose $n = 2m + 1$. Our goal is to deform $i_{H_1}(\mathbb{C}P^{n-1})$ to $i_G(M_G)$.

Let

$$S(\theta) = \left[\begin{array}{cc|ccc} 1 & 0 & & 0 & \\ 0 & 1 & & & \\ \hline & & T(\theta) & & \\ & & & T(\theta) & 0 \\ & 0 & 0 & \ddots & \\ & & & & T(\theta) \end{array}\right] \quad \begin{array}{l}(m \text{ copies of} \\ T(\theta) \text{ on} \\ \text{diagonal})\end{array}$$

From equation (3.8.1) it follows that

$$S(\pi/2)([w_1:w_2:\ldots:w_n]) = \{w_1:-\bar{w}_3:\bar{w}_2:\ldots:-\bar{w}_n:\bar{w}_{n-1}\}. \qquad (3.8.3)$$

Hence, as a mapping of $\widetilde{G}_2\mathbb{R}^{2n}$ to itself,

$$S(\pi/2)(i_{H_1}(\mathbb{C}P^{n-1})) = i_G(M_G).$$

In local coordinates on $\mathbb{C}P^{n-1}$, from (3.8.3), we get

$$\varphi\cdot S(\pi/2)\cdot i_{H_1}\cdot \psi_H^{-1}(x_1,y_1,\ldots,x_{n-1},y_{n-1}) = \begin{bmatrix} -x_2 & y_2 \\ y_2 & x_2 \\ x_1 & -y_1 \\ -y_1 & -x_1 \\ \vdots & \vdots \\ -x_n & y_n \\ y_n & x_n \\ x_{n-1} & -y_{n-1} \\ -y_{n-1} & -x_{n-1} \end{bmatrix} \qquad (3.8.4)$$

From the claim we know the intersection with

$i_{H_1}(\mathbb{C}P^{n-1})$ is transverse at P, $(x_1,y_1,\ldots,x_{n-1},y_{n-1}) = (0,\ldots,0)$, so it remains to compute the oriented intersection number. We compute the differential of (3.8.2) and (3.8.4) at 0. It is sufficient to examine the case $n = 3$.

$$d(\varphi\cdot i_{H_1}\cdot \psi_H^{-1})_0 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{bmatrix},$$

$$d(\varphi\cdot S(\pi/2)\cdot i_{H_1}\cdot \psi_H^{-1})_0 = \begin{bmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{bmatrix}.$$

If we juxtapose these two matricies and calculate the determinate of the resulting 8×8 matrix we get a positive number. Hence the direct sum of the orientations under these two embeddings is consistent with our original choice of orientation on $\tilde{G}_2\mathbb{R}^{2n}$. So for n odd

$i_{H_{1*}}([\mathbb{C}P^{n-1}])\cdot i_{H_{1*}}([\mathbb{C}P^{n-1}]) = +1$. QED.

We now have sufficient information to compute the matrix of the bilinear form q_n :

$$q_n =$$

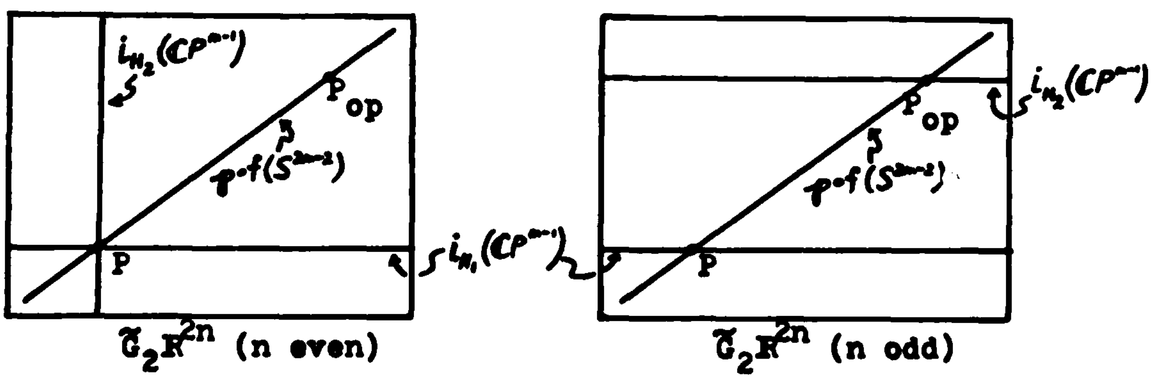
$$\begin{bmatrix} (p\cdot f)_*([S^{2n-2}])\cdot(p\cdot f)_*([S^{2n-2}]) & i_{H_{1_*}}([\mathbb{C}P^{n-1}])\cdot(p\cdot f)_*([S^{2n-2}]) \\ (p\cdot f)_*([S^{2n-2}])\cdot i_{H_{1_*}}([\mathbb{C}P^{n-1}]) & i_{H_{1_*}}([\mathbb{C}P^{n-1}])\cdot i_{H_{1_*}}([\mathbb{C}P^{n-1}]) \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 1 \\ 1 & n \bmod 2 \end{bmatrix} .$$

Let $i_{H_2} : \mathbb{C}P^{n-1} \longrightarrow \tilde{G}_2\mathbb{R}^{2n}$ be the following embedding:

1) if n is odd, $i_{H_2} = A\cdot i_{H_1}$,

2) if n is even, $i_{H_2} = i_G\cdot i$ (where $i : \mathbb{C}P^{n-1} \longrightarrow M_G$ is a diffeomorphism).



Let $\{\mathbb{C}P^{n-1}\}$ denote the fundamental cycle in $H_{2n-2}(\mathbb{C}P^{n-1})$ such that $i_{H_{2*}}(\{\mathbb{C}P^{n-1}\})\cdot(p\cdot f)_*([S^{2n-2}]) = +1$. (note that $P \in i_{H_2}(\mathbb{C}P^{n-1})$ if n is even, $P_{op} \in i_{H_2}(\mathbb{C}P^{n-1})$ if n is odd)

Now we compute the homology class of $i_{H_{2*}}(\{\mathbb{C}P^{n-1}\})$. Suppose $i_{H_{2*}}(\{\mathbb{C}P^{n-1}\}) = (a,b)$.

1) If n is odd then since $i_{H_{2*}}(\{\mathbb{C}P^{n-1}\})\cdot i_{H_{1*}}([\mathbb{C}P^{n-1}]) = 0$ we have

$$(a\ b)\begin{bmatrix}2 & 1\\ 1 & 1\end{bmatrix}\begin{bmatrix}0\\ 1\end{bmatrix} = 0$$

so $a + b = 0$. Also $i_{H_{2*}}(\{\mathbb{C}P^{n-1}\})\cdot(p\cdot f)_*([S^{2n-2}]) = +1$ gives:

$$(a\ b)\begin{bmatrix}2 & 1\\ 1 & 1\end{bmatrix}\begin{bmatrix}1\\ 0\end{bmatrix} = +1$$

so $2a + b = 1$. These two equations together imply $i_{H_{2*}}(\{\mathbb{C}P^{n-1}\}) = (1,-1)$.

2) If n is even, then an argument similar to that at the end of LEMMA 3.8 shows that $i_{H_{1*}}([\mathbb{C}P^{n-1}])\cdot i_{H_{2*}}(\{\mathbb{C}P^{n-1}\}) = +1$. Therefore

$$(a\ b)\begin{bmatrix}2 & 1\\ 1 & 0\end{bmatrix}\begin{bmatrix}0\\ 1\end{bmatrix} = +1$$

so $a = 1$. As in the odd case, we also get $2a + b = 1$, so for n even we conclude $i_{H_{2_*}}([\mathbb{C}P^{n-1}]) = (1,-1)$.

We have finally assembled enough information to complete the first step of the program we set out at the beginning of the section, namely to show $i_{F_*}(H_*(M_F)) = i_{H_{j_*}}(H_*(\mathbb{C}P^{n-1}))$ for j either 1 or 2. We may and shall assume, without loss of generality, that $P \in i_F(M_F)$. By LEMMA 3.7, $i_F(M_F)$ is transverse to $p\cdot f(S^{2n-2})$, so let $[M_F]$ denote the fundamental cycle in $H_{2n-2}(M_F)$ determined by the orientation such that

$$i_{F_*}([M_F])\cdot(p\cdot f)_*([S^{2n-2}]) = +1 .$$

There are now two possibilities, either

$$\langle i_{F_*}([M_F]), e(B)^{n-1}\rangle = +1 \quad \text{or} \quad -1 .$$

If we let $i_{F_*}([M_F]) = (a,b)$ then

$$b = \langle i_{F_*}([M_F]), e(B)^{n-1}\rangle = +1 \quad \text{or} \quad -1 .$$

1) Suppose n is even.

If $b = +1$, then

$$(a\ 1)\begin{bmatrix} 2 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = +1$$

implies $a = 0$. If $b = -1$, then

$$(a\ -1)\begin{bmatrix} 2 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = +1$$

implies $a = 1$.

2) When n is odd, similar computations with $q_n = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$ gives the identical result.

Therefore in all cases, $i_{F_*}(H_{2n-2}(M_F)) \equiv i_{H_{j_*}}(H_{2n-2}(\mathbb{C}P^{n-1}))$ for j either 1 or 2 and the particular value of j is completely given by the sign of $\langle i_{F_*}([M_F]), e(B)^{n-1}\rangle$.

In the lower dimensional cases we turn to the Gysin sequences of the Stiefel bundle B, and the fibration F. Since the embedding $i_F\colon M_F \longrightarrow \tilde{G}_2\mathbb{R}^2$ is covered by a bundle map of total spaces, by functoriality of the Gysin sequences we have the commutative diagram:

$$\begin{array}{ccccccc}
\to H^j(V_2\mathbb{R}^{2n}) \to & H^{j-1}(\tilde{G}_2\mathbb{R}^{2n}) & \xrightarrow{Ue(B)} & H^{j+1}(\tilde{G}_2\mathbb{R}^{2n}) & \to & H^{j+1}(V_2\mathbb{R}^{2n}) \to \\
\downarrow i_F^* & \downarrow i_F^* & & \downarrow i_F^* & & \downarrow i_F^* \\
\to H^j(S^{2n-1}) \longrightarrow & H^{j-1}(M_F) & \xrightarrow{Ue(F)} & H^{j+1}(M_F) & \longrightarrow & H^{j+1}(S^{2n-1}) \to
\end{array}$$

Now $H^*(M_F) \simeq \mathbb{Z}[e(F)]/(e(F)^n)$ hence for $j < 2n-2$ we have zeros on the right and left sides hence we obtain $i_{F_*}(H_j(M_F)) \simeq i_{H_{1_*}}(H_j(\mathbb{C}P^{n-1})) \simeq i_{H_{2_*}}(H_j(\mathbb{C}P^{n-1}))$. Thus the compilation of homology data portion of our proof has been completed and we now show that this information is sufficient to construct the desired homotopy g.

To be specific, for the remainder of the proof we assume $i_{F_*}([M_F]) = (0,1)$. So our goal is to construct a homotopy $g : I \times \mathbb{C}P^{n-1} \to \tilde{G}_2\mathbb{R}^{2n}$ with $g_0 = i_{H_1}$ and $g_1 = i_F \cdot h_F$ where $h_F: \mathbb{C}P^{n-1} \to M_F$ is a homotopy equivalence. From now on we drop the subscript 1 from the fibration H_1 and let H denote the standard Hopf fibration.

The homotopy equivalence h_F can be constructed very explicitly by first constructing a map $h_F: S^{2n-1} \to S^{2n-1}$ sending Hopf fibres to fibres of the fibration F. h_F may be constructed so that it preserves the orientations on the fibres of the two fibrations or reverses them. Hence we may and shall assume that $h_F^*(e(F)) = e(H)$. This implies

1) If $S^2 \simeq \mathbb{C}P^1 \subseteq \mathbb{C}P^{n-1}$ represents a generator of $H_2(\mathbb{C}P^{n-1})$ then $i_{H_*}(S^2) = (i_F \cdot h_F)_*(S^2)$ (since $\mathbb{C}P^1$ is dual to $e(H)$).

2) $i_{H_*}([\mathbb{C}P^{n-1}]) = (i_F \cdot h_F)_*([\mathbb{C}P^{n-1}])$.

Our plan is to construct the homotopy g, in a stepwise manner over $\mathbb{C}P^1 \subseteq \mathbb{C}P^2 \subseteq \ldots \subseteq \mathbb{C}P^{n-1}$, using the homology data to conclude that any possible obstructions to this procedure vanish.

NOTATION AND CONVENTIONS.

1) $I \times S^{2k-1} = \{(x_1,\ldots,x_{2k+1}) : \Sigma_{i=2}^{2k+1} x_i^2 = 1 \; , \; 0 \le x_1 \le 1\}$

$$U_0 = \{(x_1,\ldots,x_{2k+1}) : \Sigma_{i=2}^{2k+1} x_i^2 < 1 \; , \; x_1 = 0\}$$

$$U_1 = \{(x_1,\ldots,x_{2k+1}) : \Sigma_{i=2}^{2k+1} x_i^2 < 1 \; , \; x_1 = 1\}$$

2) $\mathcal{S}^{2k} = U_0 \cup (I \times S^{2k-1}) \cup U_1 \simeq S^{2k}$

3) $B^{2k+1} = I \times \bar{U}_0 = \{(x_1,\ldots,x_{2k+1}) : \Sigma_{i=2}^{2k+1} x_i^2 \le 1 \; ,$
$0 \le x_1 \le 1\}$

4) $\mathbb{C}P^k = \{(x_1,\ldots,x_{2k}) : \Sigma_{i=1}^{2k} x_i^2 \le 1\} \cup_{\pi_{k-1}} \mathbb{C}P^{k-1}$ (where

$\pi_{k-1}: \; S^{2k-1} \longrightarrow \mathbb{C}P^{k-1}$ is the Hopf projection).

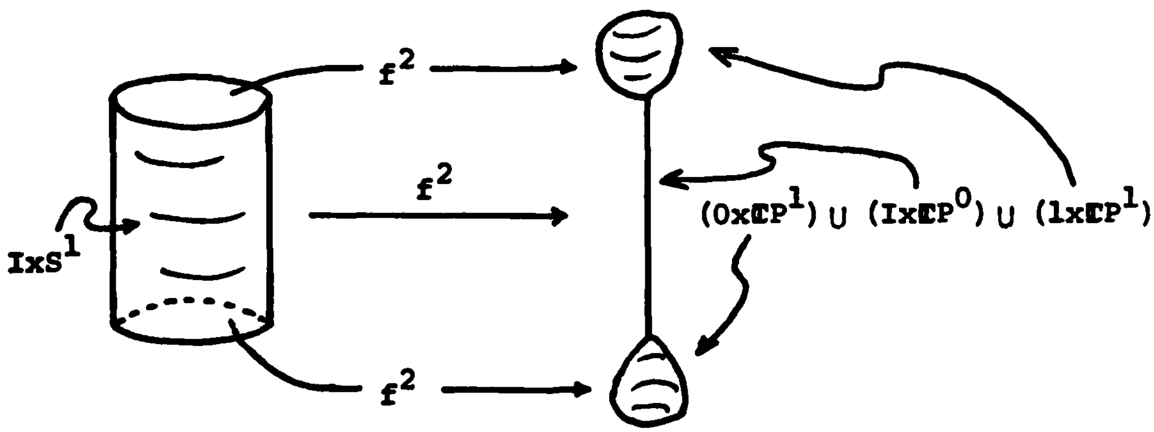
5) $f^{2k}: \; S^{2k} \longrightarrow (0 \times \mathbb{C}P^{k}) \cup (I \times \mathbb{C}P^{k-1}) \cup (1 \times \mathbb{C}P^{k})$ is given by

$$f^{2k}(x) = f^{2k}(x_1, \ldots, x_{2k+1}) = \begin{cases} (x_1, (x_2, \ldots, x_{2k+1})) & x \in U_0 \text{ or } U_1 \\ (x_1, \pi_{k-1}(x_2, \ldots, x_{2k+1})) & x \in I \times S^{2k-1} \end{cases}$$

6) $f^{2k+1}: \; B^{2k+1} \longrightarrow I \times \mathbb{C}P^{k}$

$$f^{2k+1}(x) = f^{2k+1}(x_1, \ldots, x_{2k+1}) = \begin{cases} (x_1, (x_2, x_3, \ldots, x_{2k+1})) & x \in \operatorname{Int} B^{2k+1} \\ f^{2k}(x_1, \ldots, x_{2k+1}) & x \in \partial B^{2k+1} \end{cases}$$

These rather cumbersome looking definitions can best be summed up by the following picture for $k = 1$:



LEMMA 3.10. If $\bar{g} : B^{2k+1} \to X$ is any map with $\bar{g}|\partial B^{2k+1} = \tilde{g}\cdot f^{2k}$ where

$\tilde{g} : (0 \times \mathbb{C}P^k) \cup (I \times \mathbb{C}P^{k-1}) \cup (1 \times \mathbb{C}P^k) \to X$, then there exists $g : I \times \mathbb{C}P^k \to X$ such that $\bar{g} = g\cdot f^{2k+1}$.

$$\begin{tikzcd} & I \times \mathbb{C}P^k \arrow[dr, dashed, "g"] & \\ B^{2k+1} \arrow[ur, "f^{2k+1}"] \arrow[rr, "\bar{g}"] & & X \end{tikzcd}$$

PROOF: Define

$$g(t,x) = \begin{cases} \bar{g}(t,x) & \text{if } x \in \mathbb{C}P^k - \mathbb{C}P^{k-1} \\ \tilde{g}(t,x) & \text{if } x \in \mathbb{C}P^{k-1} \end{cases}$$

QED.

From the homotopy sequence of the Stiefel bundle B, we get the exact sequence:

$$\pi_j(S^1) \to \pi_j(V_2\mathbb{R}^{2n}) \to \pi_j(\tilde{G}_2\mathbb{R}^{2n}) \to \pi_{j-1}(S^1).$$

So from our assumption $n > 2$, we conclude from the homotopy information of S^1 and $V_2\mathbb{R}^{2n}$ that: $\pi_2(\tilde{G}_2\mathbb{R}^{2n}) \simeq \pi_{2n-2}(\tilde{G}_2\mathbb{R}^{2n}) \simeq \mathbb{Z}$, $\pi_j(\tilde{G}_2\mathbb{R}^2) = 0$ for all other $j < 2n-2$. So by the Hurewicz Isomorphism Theorem,

$\pi_2(\tilde{G}_2\mathbb{R}^{2n}) \simeq H_2(\tilde{G}_2\mathbb{R}^{2n})$ and since $i_H(\mathbb{C}P^1)$ is homologous to $i_F \cdot h_F(\mathbb{C}P^1)$ there exists a homotopy

$g^1\colon\ I \times \mathbb{C}P^1 \longrightarrow \tilde{G}_2\mathbb{R}^{2n}$ with $g_0{}^1 = i_H|\mathbb{C}P^1$ and $g_1{}^1 = i_F \cdot h_F|\mathbb{C}P^1$.

Define

$$j^2\colon\ (0 \times \mathbb{C}P^2) \cup (I \times \mathbb{C}P^1) \cup (1 \times \mathbb{C}P^2) \longrightarrow \tilde{G}_2\mathbb{R}^{2n}$$

$$j^2(0,x) = i_H(x)$$

$$j^2(t,x) = g_t{}^1(x)$$

$$j^2(1,x) = i_F \cdot h_F(x).$$

Define

$$\bar{g}^2|S^4 = j^2 \cdot f^4\colon\ S^4 \longrightarrow \tilde{G}_2\mathbb{R}^{2n}\ .$$

If $n > 3$ then $\pi_4(\tilde{G}_2\mathbb{R}^{2n}) = 0$ and $\bar{g}^2|S^4$ extends to a map $\bar{g}^2\colon\ B^5 \longrightarrow \tilde{G}_2\mathbb{R}^{2n}$ where $\bar{g}^2|\partial B^5 = \bar{g}^2|S^4 = j^2 \cdot f^4$. By LEMMA 3.10, $\bar{g}^2 = g^2 \cdot f^5$ where $g^2\colon\ I \times \mathbb{C}P^2 \longrightarrow \tilde{G}_2\mathbb{R}^{2n}$. Since $g^2|(0 \times \mathbb{C}P^2) = j^2|(0 \times \mathbb{C}P^2) = i_H|\mathbb{C}P^2$ and $g^2|(1 \times \mathbb{C}P^2) = j^2|(1 \times \mathbb{C}P^2) = i_F \cdot h_F|\mathbb{C}P^2$, g^2 is the desired extension of the homotopy g^1 .

If $2 < m < 2n-2$ then $\pi_m(\tilde{G}_2\mathbb{R}^{2n}) = 0$ so its clear we can continue this stepwise extension process without obstruction up to a homotopy

$$g^{n-2}: I \times \mathbb{C}P^{n-2} \longrightarrow \tilde{G}_2\mathbb{R}^{2n} .$$

Trying to continue we construct

$$j^{n-1}: (0 \times \mathbb{C}P^{n-1}) \cup (I \times \mathbb{C}P^{n-2}) \cup (1 \times \mathbb{C}P^{n-1}) \longrightarrow \tilde{G}_2\mathbb{R}^{2n}$$

as above, which leads to a map $\bar{g}^{n-1}|S^{2n-2}: S^{2n-2} \longrightarrow \tilde{G}_2\mathbb{R}^{2n}$, $\bar{g}^{n-1}|S^{2n-2} = j^{n-1} \cdot f^{2n-2}$. $\bar{g}^{n-1}|S^{2n-2}$ represents an element of $\pi_{2n-2}(\tilde{G}_2\mathbb{R}^{2n}) \simeq \mathbb{Z}$ and if this is the zero element then we can proceed exactly as in the first stage, extending to $\bar{g}^{n-1}: B^{2n-1} \longrightarrow \tilde{G}_2\mathbb{R}^{2n}$ with $\bar{g}^{n-1} = g^{n-1} \cdot f^{2n-1}$ by LEMMA 3.10. But g^{n-1} is a map from $I \times \mathbb{C}P^{n-1}$ to $\tilde{G}_2\mathbb{R}^{2n}$ with $g^{n-1}|(0 \times \mathbb{C}P^{n-1}) = i_H$ and $g^{n-1}|(1 \times \mathbb{C}P^{n-1}) = i_F \cdot h_F$ so setting $g = g^{n-1}$ gives the desired homotopy and completes the proof of THEOREM A.

Hence it remains to prove

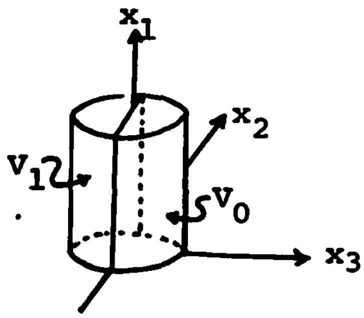
LEMMA 3.11. $\bar{g}^{n-1}|S^{2n-2}$ represents the 0 element of $\pi_{2n-2}(\tilde{G}_2\mathbb{R}^{2n})$.

PROOF: We write g for the map $\bar{g}^{n-2}|S^{2n-2}$. Let

$$V_0 = \{(x_1,\ldots,x_{2n-1}) : \Sigma_{i=2}^{2n-1} x_i^2 = 1 \ , \ x_2 \geq 0 \ , \ 0 \leq x_1 \leq 1\}$$

$$V_1 = \{(x_1,\ldots,x_{2n-1}) : \Sigma_{i=2}^{2n-1} x_i^2 = 1 \ , \ x_2 \leq 0 \ , \ 0 \leq x_1 \leq 1\}$$

$S^{2n-2} = \bar{U}_0 \cup V_0 \cup V_1 \cup \bar{U}_1$



Let $j_i : \bar{U}_i \longrightarrow \mathbb{C}P^{n-1}$ be the identification $\mathbb{C}P^{n-1} = \bar{U}_i \cup_{\pi_{n-2}} \mathbb{C}P^{n-2}$ for $i = 0,1$. Orient the cell $\bar{U}_0$ so that j_0 is orientation preserving (recall the original orientation on $\mathbb{C}P^{n-1}$ to define the fundamental cycle $[\mathbb{C}P^{n-1}]$). Now orient the remaining cells V_0, V_1, $\bar{U}_1$ such that the induced orientations on the boundaries cancel and we get a fundamental cycle, $[S^{2n-2}] = \bar{U}_0 + V_0 + V_1 + \bar{U}_1$. Therefore on the chain level, $g_\#(S^{2n-2}) = g_\#(\bar{U}_0) + g_\#(V_0) + g_\#(V_1) + g_\#(\bar{U}_1)$. Now $g(V_i)$, $i = 0,1$, factors through $I \times \mathbb{C}P^{n-2}$ which is $2n-3$ dimensional, hence considered as $2n-2$ singular chains they are homologous to 0. So, if $\langle\ \rangle$ denotes the homology class of a chain, $g_*([S^{2n-2}]) = \langle g(\bar{U}_0) + g(\bar{U}_1)\rangle = \langle i_H \cdot j_0(\bar{U}_0) + i_F \cdot h_F \cdot j_1(\bar{U}_1)\rangle = i_{H_*}([\mathbb{C}P^{n-1}]) + i_F \cdot h_F(-[\mathbb{C}P^{n-1}]) = (0,1) - (0,1) = 0$ (note that the orientation on $\bar{U}_1$ is opposite to that on $\bar{U}_0$).

For $n \geq 3$ we have the commutative diagram:

(exact homotopy sequence of Stiefel bundle)

$$\begin{tikzcd}[column sep=small]
\to 0 = \pi_{2n-2}(S^1) \arrow[r] & \pi_{2n-2}(V_2\mathbb{R}^{2n}) \arrow[r, "\simeq"] \arrow[d, "\rho_V"] & \pi_{2n-2}(\tilde{G}_2\mathbb{R}^{2n}) \arrow[r] \arrow[d, "\rho_G"] & \pi_{2n-3}(S^1) = 0 \\
\to 0 = H_{2n-3}(\tilde{G}_2\mathbb{R}^{2n}) \arrow[r] & H_{2n-2}(V_2\mathbb{R}^{2n}) \arrow[r, "p_*"] & H_{2n-2}(\tilde{G}_2\mathbb{R}^{2n}) &
\end{tikzcd}$$

(exact homology Gysin sequence of Stiefel bundle)

Where ρ_V and ρ_G are Hurewicz homomorphisms and ρ_V is an ismorphism. From exactness of the bottom row, p_* is a monomorphism, hence by commutativity we conclude ρ_G is a monomorphism. But $\rho_G(\langle g\rangle) = g_*([S^{2n-2}]) = 0$ hence $\langle g\rangle = 0$ in $\pi_{2n-2}(\tilde{G}_2\mathbb{R}^{2n})$. QED.

SECTION IV

Every great circle fibration of S^3 has some orthogonal pair of fibres ([G-W], Theorem C). The proof of this fact, along with other results in [G-W] show that it is equivalent to the Borsak-Ulam Theorem for maps of S^2 to $\mathbb{R}^2$. Since the Borsuk-Ulam Theorem is valid for all dimensions, it seems natural to conjecture that the same result holds for great 3-sphere fibrations of S^7 and great 7-sphere fibrations of S^{15}. In this section we demonstrate that this is not the case by providing an explicit example of a great 3-sphere fibration of S^7 with no orthogonal pairs of fibres. A completely analogous approach provides an example of a great 7-sphere fibration of S^{15} with no orthogonal pairs of fibres.

NOTATION AND CONVENTIONS.

1) $\{e_1,\ldots,e_8\}$ is the standard orthonormal basis for $\mathbb{R}^8$.
2) P_0 will be the 4-plane, $P_0 = \mathrm{span}(e_1,e_2,e_3,e_4)$.
3) P_∞ will be the 4-plane,
   $P_\infty = \mathrm{span}(e_5,e_6,e_7,\frac{\sqrt{2}}{2}e_4+\frac{\sqrt{2}}{2}e_8)$.
4) We identify the quaternions $\mathbb{H}$ with

$\mathbb{R}^4$ in the usual way,

$a = a_1 + a_2 i + a_3 j + a_4 k \longrightarrow (a_1, a_2, a_3, a_4)$.

5) We also identify $\mathbb{H}$ with a 4-dimensional linear subspace of $GL(4,\mathbb{R}) \cup \{0\}$, via the "left multiplication map":

$$a = a_1 + a_2 i + a_3 j + a_4 k \longrightarrow L_a = \begin{bmatrix} a_1 & -a_2 & -a_3 & -a_4 \\ a_2 & a_1 & -a_4 & a_3 \\ a_3 & a_4 & a_1 & -a_2 \\ a_4 & -a_3 & a_2 & a_1 \end{bmatrix}$$

This map gives an algebra isomorphism of $\mathbb{H}$ onto its image.

6) We view the Hopf fibration $S^3 \longrightarrow S^7 \longrightarrow S^4$ as the "graphs" of the left multiplication map given in (5). Specifically, the Hopf fibres are given by the family of 4-planes

$$Q_a = \{(w, aw) : w \in \mathbb{H}\} \quad \text{for all} \quad a \in \mathbb{H}$$

along with the 4-plane $Q_\infty = Q_0^{\perp}$.

From this, and the matrix L_a, we get (where $a = a_1 + a_2 i + a_3 j + a_4 k$)

$$\begin{aligned} Q_a = \operatorname{span}(e_1 &+ a_1e_5 + a_2e_6 + a_3e_7 + a_4e_8\,, \\ e_2 &- a_2e_5 + a_1e_6 + a_4e_7 - a_3e_8\,, \\ e_3 &- a_3e_5 - a_4e_6 + a_1e_7 + a_2e_8\,, \\ e_4 &- a_4e_5 + a_3e_6 - a_2e_7 + a_1e_8). \end{aligned}$$

We begin by exhibiting a fibration of S^7 by great 3-spheres with precisely one pair of orthogonal fibres. Naturally the next step would be to try to perturb this fibration in a neighborhood of one member of the orthogonal pair hoping that this perturbation introduces no new orthogonal pairs. We show that the orthogonal pair in this fibration is the only pair of fibres that satisfies a weaker necessary condition for orthogonality. This allows us to conclude that if we are judicious in choosing the perturbation we can guarantee we introduce no new orthogonal pairs while still destroying the orthogonality of the original pair.

Let $T \in GL(8,\mathbb{R})$ be the linear transformation given by:

$$T(e_i) = e_i \quad 1 \le i \le 7\,, \quad T(e_8) = \frac{\sqrt{2}}{2}\,e_4 + \frac{\sqrt{2}}{2}\,e_8\,.$$

If we apply T to the 4-planes of the Hopf fibration we get:

$$T(Q_a) = P_a = \mathrm{span}(e_1 + \tfrac{\sqrt{2}}{2} a_4 e_4 + a_1 e_5 + a_2 e_6 + a_3 e_7 + \tfrac{\sqrt{2}}{2} a_4 e_8,$$
$$e_2 - \tfrac{\sqrt{2}}{2} a_3 e_4 - a_2 e_5 + a_1 e_6 + a_4 e_7 - \tfrac{\sqrt{2}}{2} a_3 e_8,$$
$$e_3 + \tfrac{\sqrt{2}}{2} a_2 e_4 - a_3 e_5 - a_4 e_6 + a_1 e_7 + \tfrac{\sqrt{2}}{2} a_2 e_8,$$
$$e_4 + \tfrac{\sqrt{2}}{2} a_1 e_4 - a_4 e_5 + a_3 e_6 - a_2 e_7 + \tfrac{\sqrt{2}}{2} a_1 e_8)$$

$= \mathrm{span}(f_1(a), f_2(a), f_3(a), f_4(a))$ where

$f_i\colon \mathbb{R}^4 \to \mathbb{R}^8$ is the ith spanning vector of

P_a shown here, $1 \le i \le 4$.

$T(Q_\infty) = P_\infty$.

So the family of 4-planes $\{P_a\colon a \in \mathbb{H} \cup \{\infty\}\}$ gives us a new great 3-sphere fibration of S^7 equivalent to the Hopf fibration.

Let $F = (f_1, f_2, f_3, f_4)\colon \mathbb{R}^4 \to \mathbb{R}^8 \times \mathbb{R}^8 \times \mathbb{R}^8 \times \mathbb{R}^8$.

Note that

$$P_1 = \mathrm{span}\, F(1,0,0,0) = \mathrm{span}(e_1 + e_5, e_2 + e_6, e_3 + e_7, (1 + \tfrac{\sqrt{2}}{2}) e_4 + \tfrac{\sqrt{2}}{2} e_8)$$

$$P_{-1} = \mathrm{span}\, F(-1,0,0,0) = \mathrm{span}(e_1 - e_5, e_2 - e_6, e_3 - e_7, (1 - \tfrac{\sqrt{2}}{2}) e_4 - \tfrac{\sqrt{2}}{2} e_8)$$

Chacking pairwise dot products between spanning vectors for P_1 and P_{-1} we see that $P_1 = P_{-1}^{\perp}$.

We will show that (P_1, P_{-1}) are the only orthogonal pair of fibres in this fibration by showing that in fact a weaker orthogonality property holds only for the fibres P_1 and P_{-1} among all fibres lying over $\mathbb{H}$ (excluding P_∞). For $a \in \mathbb{H}_1$ let R_a denote the 3-plane in $\mathbb{R}^8$, $R_a = \text{span}(f_2(a), f_3(a), f_4(a)) \subseteq P_a$. $R_a^{\perp}$ is a 5-plane which may or may not contain a fibre of the fibration (checking codimensions it can clearly contain at most one fibre).

LEMMA 4.1. $R_a^{\perp}$ contains a fibre of the fibration, P_b, if and only if $a = \pm 1$ (in which case $b = \mp 1$).

PROOF: $R_a^{\perp} \supseteq P_b$ if and only if each of the 3 vectors spanning R_a is orthogonal to all four spanning vectors of P_b. Therefore we compute:

$$0 = f_2(a) \cdot f_1(b) = -a_2b_1 + a_1b_2 + a_4b_3 - a_3b_4$$

$$0 = f_2(a) \cdot f_2(b) = 1 + a_2b_2 + a_1b_1 + a_4b_4 + a_3b_3$$

$$0 = f_2(a) \cdot f_3(b) = a_2b_3 - a_1b_4 + a_4b_1 - a_3b_2$$

$$0 = f_2(a) \cdot f_4(b) = a_2b_4 + a_1b_3 - a_4b_2 - a_3b_1 - \frac{\sqrt{2}}{2} a_3$$

$$\vdots$$

$$0 = f_4(a)\cdot f_4(b) = a_4b_4 + a_3b_3 + a_2b_2 + \frac{1}{2}a_1b_1 + $$
$$+ \left(1+\frac{\sqrt{2}}{2}a_1\right)\left(1+\frac{\sqrt{2}}{2}b_1\right)$$
$$= a_4b_4 + a_3b_3 + a_2b_2 + \left(\frac{\sqrt{2}}{2}+a_1\right)b_1 + $$
$$+ \left(1+\frac{\sqrt{2}}{2}a_1\right).$$

This system of 12 equations is equivalent to the matrix equation:

$$\underbrace{\begin{bmatrix} -a_2 & a_1 & a_4 & -a_3 \\ -a_3 & -a_4 & a_1 & a_2 \\ -a_4 & a_3 & -a_2 & \frac{\sqrt{2}}{2}+a_1 \end{bmatrix}}_{A}\underbrace{\begin{bmatrix} b_1 & -b_2 & -b_3 & -b_4 \\ b_2 & b_1 & -b_4 & b_3 \\ b_3 & b_4 & b_1 & -b_2 \\ b_4 & -b_3 & b_2 & b_1 \end{bmatrix}}_{B} = \underbrace{\begin{bmatrix} 0 & -1 & 0 & \frac{\sqrt{2}}{2}a_3 \\ 0 & 0 & -1 & -\frac{\sqrt{2}}{2}a_2 \\ 0 & 0 & 0 & -\left(1+\frac{\sqrt{2}}{2}a_1\right) \end{bmatrix}}_{C}$$

Suppose $a = a_1 + a_2i + a_3j + a_4k$ and $b = b_1 + b_2i + b_3j + b_4k$ are a pair of quaternions whose entries satisfy the above system. If $\bar{a}\cdot b = c_1 + c_2i + c_3j + c_4k$ then its easy to check that:

row 1 of $A\cdot$col 1 of $B = c_2$

row 2 of $A\cdot$col 2 of $B = c_1$

$$\text{row 1 of } A\cdot\text{col 3 of } B = -C_4$$

$$\text{row 2 of } A\cdot\text{col 1 of } B = C_3'$$

So comparing with the entries in C we conclude that $\bar{a}\cdot b = -1$.

Also we have

$$0 = \text{row 3 of } A\cdot\text{col 1 of } B = c_4 + \frac{\sqrt{2}}{2}b_4 = \frac{\sqrt{2}}{2}b_4 \quad \text{hence} \quad b_4 = 0$$

$$0 = \text{row 3 of } A\cdot\text{col 2 of } B = -c_3 - \frac{\sqrt{2}}{2}b_3 = -\frac{\sqrt{2}}{2}b_3 \quad \text{hence} \quad b_3 = 0$$

$$0 = \text{row 3 of } A\cdot\text{col 3 of } B = c_2 + \frac{\sqrt{2}}{2}b_2 = \frac{\sqrt{2}}{2}b_2 \quad \text{hence} \quad b_2 = 0$$

Therefore $b \in \mathbb{R}$ so $\bar{a}\cdot b = -1$ implies $a \in \mathbb{R}$, $a = a_1$, $b = b_1$. Finally, $-(1+\frac{\sqrt{2}}{2}a_1) =$ row 3 of $A\cdot$col 4 of $B =$ $= (\frac{\sqrt{2}}{2}+a_1)b_1$ so $-1 - \frac{\sqrt{2}}{2}a_1 = \frac{\sqrt{2}}{2}b_1 + a_1b_1 = \frac{\sqrt{2}}{2}b_1 - 1$ hence $-\frac{\sqrt{2}}{2}a_1 = \frac{\sqrt{2}}{2}b_1$ and $a_1 = -b_1$. $a_1 = -b_1$ and $a_1b_1 = -1$ together yield $a = \pm 1$ and $b = \mp 1$. QED.

COROLLARY 4.2. The only orthogonal pair of fibres in this fibration are P_1 and P_{-1} .

PROOF: We've already shown P_1 and P_{-1} are an orthogonal pair. P_∞ is not a member of an orthogonal pair since $e_1 \in P_\infty^\perp \cap P_0$ but $P_\infty^\perp \neq P_0$. This implies $P_\infty^\perp$ is not a fibre.

If $P_a{}^{\perp} = P_b$ for $a , b \in \mathbb{H}$, then clearly $R_a{}^{\perp} \supseteq P_a{}^{\perp} = P_b$. By the LEMMA this is only possible for $a = \pm 1$. QED.

Now we proceed to perturb this fibration in a neighborhood of the fibre P_1. We perturb it in such a way that the new fibre over 1, P_1' is no longer orthogonal to P_{-1}, and so that we can still use LEMMA 4.1 to conclude we have introduced no new orthogonal pairs.

Let B_ϵ be the closed ball about $(1,0,0,0) \in \mathbb{R}^4$, where we choose $0 < \epsilon < \frac{1}{2}$ small enough so that for all pairs $a , b \in B_\epsilon$, R_a and R_b are not orthogonal. Since f_2, f_3 and f_4 are continuous we can certainly find such an ϵ (actually any $\epsilon < \frac{1}{2}$ will work).

REMARK 4.3. Note that $f_1(p)$ is <u>not</u> orthogonal to P_{-1} for $(1,0,0,0) \neq p \in B_\epsilon$. This follows since all vectors orthogonal to P_{-1} lie in P_1 by COROLLARY 4.2 $(P_{-1}{}^{\perp} = P_1)$, but for $p \neq (1,0,0,0)$, $f_1(p) \notin P_1$.

Since we still want to apply LEMMA 4.1 to our new fibration we don't want to perturb the 3 vectors $f_2(a)$, $f_3(a)$ and $f_4(a)$ lying in the plane P_a. We only want to move the vector $f_1(a)$.

Define $G : \mathbb{R}^4 \times S^3 \longrightarrow \mathbb{R}^8$

$$G(a_1,\ldots,a_4,b_1,\ldots,b_4) = \sum_{i=1}^{4} b_i f_i(a)$$

where $b_1^2 + b_2^2 + b_3^2 + b_4^2 = 1$. $(a,b) \longrightarrow G(a,b)/\|G(a,b)\|$ is just a parameterization of our great 3-sphere fibration of S^7 over the open set $\mathbb{R}^4 \simeq \mathbb{H}^4 \subseteq \mathbb{H}^4 \cup \{\infty\}$. Let $\tilde{G} = G|B_\epsilon \times S^3$. $\tilde{G}$ is a diffeomorphism from the compact set $B_\epsilon \times S^3$ onto its image.

For any $\delta > 0$, there exists a diffeomorphism $g : \mathbb{R}^4 \longrightarrow \mathbb{R}^4$ that is the identity on the exterior of $B_{\epsilon/2}$, maps $(1,0,0,0)$ to $(1+\frac{\epsilon}{n},0,0,0)$ for n sufficiently large so that

$$\sup_{\substack{i,j \\ x\in\mathbb{R}^4}} \left|\frac{\partial g_i}{\partial x_j}(x) - \delta_{ij}\right|$$

is sufficiently small so that the map $\tilde{G}' : B_\epsilon \times S^3 \longrightarrow \mathbb{R}^8$,

$$\tilde{G}'(a_1,\ldots,a_4,b_1,\ldots,b_4) = b_1 f_1 \cdot g(a) + \sum_{i=2}^{4} b_i f_i(a)$$

satisfies $\|\tilde{G}'(x) - \tilde{G}(x)\|_{C^1} < \delta$ (where $\|\ \|_{C^1}$ is the C^1 norm). Now $\tilde{G}$ is an embedding so for sufficiently small C^1 perturbations it remains an embedding ([HI], Chap 2, Lemma 1.3). Assume g and n are chosen such that $\widetilde{G}'$ is an embedding. Let G' denote the extension of $\widetilde{G}'$ to a map from $\mathbb{R}^4 \times S^3 \longrightarrow \mathbb{R}^8$ by setting $G' = G$ on the exterior of $B_{\epsilon/2}$. Since G' is a diffeomorphism from $\mathbb{R}^4 \times S^3$ onto its image we conclude that the new family of 4-planes

$$P_a' = \text{span}(f_1 \cdot g(a), f_2(a), f_3(a), f_4(a)), \ a \in \mathbb{H}$$

along with $P_\infty' = P_\infty$ determine a smooth great 3-sphere fibration of S^7. Note that for all $a \in \mathbb{H} \cup \{\infty\}$, $a \notin B_{\epsilon/2}$, $P_a = P_a'$. For all $a \in \mathbb{H}$ define
$R_a' = \text{span}(f_2(a), f_3(a), f_4(a))$ (this is just cosmetic since trivially, $R_a' = R_a$).

Finally it remains to observe that this fibration has no orthogonal pairs of fibres. Suppose $P_a'^{\perp} = P_b'$ for some a, $b \in \mathbb{H}$ (clearly a or $b = \infty$ is impossible since $P_\infty' = P_\infty$ and $P_0' = P_0$). From COROLLARY 4.2, at least one of a or b must lie in B_ϵ (in fact $B_{\epsilon/2}$), say a. Since $R_c = R_c'$ for all $c \in \mathbb{H}$, from the initial restriction placed on ϵ it follows that $a \in B_\epsilon$ implies $b \notin B_\epsilon$, hence $P_b' = P_b$. Therefore $R_a^{\perp} = R_a'^{\perp} \supseteq P_b' = P_b$ so by LEMMA 4.1, $a = 1$ and $b = -1$. But

$f_1(1+\frac{\epsilon}{n}, 0,0,0) = f_1 \cdot g(1,0,0,0) \in P_1'$ and by REMARK 4.3 this is not orthogonal to $P_{-1} = P_{-1}'$. Hence P_1' is not orthogonal to P_{-1}' and this fibration has no orthogonal pairs.

SECTION V

In this final section we initiate a general study of great sphere fibrations of arbitrary manifolds. All fibrations in this section are assumed C^∞, over a compact C^∞ base space, and the group of all k-sphere bundles will be the orthogonal group on $\mathbb{R}^{k+1}$, $O(k+1)$.

This section is divided into three parts. In Part 1 we prove a realization theorem, Theorem B, which asserts that all reasonable k-sphere bundles can be realized as a fibration by great k-spheres by embedding the total space in S^N for N sufficiently large. As a corollary we derive what this theorem says about embedding the base space in Grassmann manifolds. The concept of a strongly injective embedding of an open set, an embedding $\varphi : U \longrightarrow \mathrm{Hom}(\mathbb{R}^m, \mathbb{R}^n)$ such that for $x \neq y$ in U, $\varphi(x)(v) \neq \varphi(y)(v)$ for all $0 \neq v \in \mathbb{R}^n$, is introduced and its relation to great sphere fibrations is discussed.

In Part 2 we examine great sphere fibrations of $S^m_{1/\sqrt{2}} \times S^n_{1/\sqrt{2}} \subseteq S^{m+n+1}$. We prove a number of general statements about the existence or non-existence of great k-sphere fibrations of $S^m_{1/\sqrt{2}} \times S^n_{1/\sqrt{2}}$ depending on k, m and n.

Finally in Part 3 we examine in detail great 3-sphere fibrations of $S^3_{1/\sqrt{2}} \times S^3_{1/\sqrt{2}} \subseteq S^7$. Using the related notion of strongly injective embeddings we completely answer the three questions about fibrations posed in Section I. In addition we relate results proved in this section to great circle fibrations of S^3.

PART 1.

THEOREM B. Let $\xi : S^k \longrightarrow E \xrightarrow{\pi} B$ be a smooth k-sphere bundle with group $O(k+1)$ over the compact base space B. Then the total space E can be smoothly embedded into S^N for N sufficiently large so that each k-sphere fibre becomes a great k-sphere in S^N.

PROOF:

1. Since the group of the bundle is $O(k+1)$ there is an associated Euclidean $(k+1)$-sphere bundle, $\xi' : \mathbb{R}^{k+1} \longrightarrow E' \longrightarrow B$, over B such that ξ is the unit sphere bundle of ξ'. By ([M-S], Lemma 5.3) there exists an integer p and a bundle map

$\tilde{f} : \xi' \longrightarrow \gamma^{k+1}\mathbb{R}^{p+1}$ where $\gamma^{k+1}\mathbb{R}^{p+1}$ is the canonical $(k+1)$-plane bundle over $G_{k+1}\mathbb{R}^{p+1}$.

$$\begin{array}{ccc} \tilde{f} : & E' & \longrightarrow E(\gamma^{k+1}\mathbb{R}^{p+1}) \\ & \downarrow \pi' & \downarrow \\ \bar{f} : & B & \longrightarrow G_{k+1}\mathbb{R}^{p+1} \end{array}$$

Now $E \subseteq E'$ and $\tilde{f}|E : E \longrightarrow S^p \subseteq \mathbb{R}^{p+1}$ so let $f = \tilde{f}|E$. Since $f(\pi^{-1}(b)) = \tilde{f}(\pi'^{-1}(b)) \cap S^p$ its clear that f(k-sphere fibre) is a great k-sphere in S^p, but while f is an embedding on each fibre, $f|\pi^{-1}(b)$, f is by no means an embedding of E in S^p.

2. If $n = 2 \cdot \dim B$, by the Whitney Embedding Theorem there exists an embedding $\psi : B \longrightarrow S^n$. Consider the map

$$(\psi \cdot \pi, f) : E \longrightarrow S^n \times S^p$$

For $a_1 \neq a_2$ in E, if $\pi(a_1) \neq \pi(a_2)$ then $\psi \cdot \pi(a_1) \neq \psi \cdot \pi(a_2)$. If $\pi(a_1) = \pi(a_2)$ then $f(a_1) \neq f(a_2)$ since f is injective on each fibre. Therefere $(\psi \cdot \pi, f)$ is an injective map from a compact space E into a Hausdorff space $S^n \times S^k$, thus it must be a homeomorphism onto its image.

3. At $a \in \pi^{-1}(x) \subseteq E$, the tangent space $T_a E$ decomposes into a direct sum, $T_a E = T_a \pi^{-1}(x) \oplus T_a \hat{B}$ where $T_a \hat{B}$ is transverse to the fibre $\pi^{-1}(x)$, and consequently maps bijectively via $d\pi$ on $T_x B$. Since $df_a(T_a \pi^{-1}(x)$ is injective and $d(\psi \cdot \pi)_a T_a \hat{B}$ is injective, it follows that $(\psi \cdot f, \pi)$ is an immersion.

4. Together the results of 2 and 3 allow us to conclude that $(\psi \cdot \pi, f)\colon E \longrightarrow S^n \times S^p$ is a smooth embedding. The following LEMMA completes the proof of THEOREM B.

LEMMA 5.1. For any pair of positive integers n and p , there exists an N and a smooth embedding of $S^n \times S^p$ into S^N , taking each submanifold $\{a\} \times S^r$, where $a \in S^n$ and S^r is a great r-subsphere of S^p , onto a great r-subsphere of S^N .

PROOF: Let $N = (p+1)(n+1) + p$ and $S^N \subseteq \mathbb{R}^{N+1}$. We show there is a smooth embedding $\varphi : S^n \longrightarrow V_{p+1}\mathbb{R}^{N+1}$ such that for $a \neq b$ in S^n the $(p+1)$-plane $\{\text{span } \varphi(a)\}$, intersects the $(p+1)$-plane $\{\text{span } \varphi(b)\}$ only at the origin.

Define $\varphi : S^n \longrightarrow \mathbb{R}^{N+1} \times \mathbb{R}^{N+1} \times \ldots \times \mathbb{R}^{N+1}$ (p+1 copies)

$$\varphi_1(x_1,\ldots,x_{n+1}) = 1/\sqrt{2}\ (1,0,\ldots,0,x_1,\ldots,x_{n+1},0,\ldots,0)$$

where x_1 is in the $p+2$nd entry.

$$\varphi_2(x_1,\ldots,x_{n+1}) = 1/\sqrt{2}\ (0,1,0,\ldots,0,0,\ldots,0,x_1,\ldots,x_{n+1}, 0,\ldots,0)$$

$\vdots$ where x_1 is the $p+n+3$rd entry.

$$\varphi_{p+1}(x_1,\ldots,x_{n+1}) = 1/\sqrt{2}\ (0,\ldots,0,1,0,\ldots,0,x_1,\ldots,x_{n+1})$$

where 1 is in the $p+1$st entry and x_1 is in the $p+pn+p+2$nd entry.

Let $\varphi = (\varphi_1,\ldots,\varphi_{p+1})$.

Suppose for $(\alpha_1,\ldots,\alpha_{p+1})$, $(\beta_1,\ldots,\beta_{p+1})$ in $\mathbb{R}^{p+1}$ we have

$$\Sigma_{i=1}^{p+1} \alpha_i\varphi_i(x) = \Sigma_{i=1}^{p+1} \beta_i\varphi_i(y) \quad \text{in} \quad \mathbb{R}^{N+1}, \text{ for } x,y \in S^n .$$

Comparing the first $k+1$ entries on each side it follows that $\alpha_i = \beta_i$, $1 \leq i \leq k+1$. Now if $\alpha_j = \beta_j \neq 0$ for some j, to be specific suppose $j = 2$, then comparing the entries from the $p+2n+2+1$ position through the $p+2n+p+1+n$ position we get $\alpha_2 x_i = \beta_2 y_i = \alpha_2 y_i$ for all $1 \leq i \leq n+1$, hence we conclude $x = y$.

If we define $\tau : S^n \times S^p \longrightarrow S^N$ by

$$\tau(a,(\alpha_1,\ldots,\alpha_{p+1})) = \Sigma_{i=1}^{p+1} \alpha_i \varphi_i(a)$$

then the above shows τ is 1-1 onto its image. As in part 2 we conclude τ is a homeomorphism onto its image. Since τ is a linear map on the second factor it clearly takes submanifolds of the form $\{a\} \times S^r$, $a \in S^n$, S^r a great r-subsphere of S^p, onto great r-subspheres of S^N. That τ is a smooth embedding is clear. QED.

Finally, $\tau\cdot(\psi\cdot\pi, f): E \longrightarrow S^N$ is an embedding that satisfies the requirements of THEOREM B. QED THM B.

COROLLARY 5.2. Given the hypothesis of THEOREM B, the base space B of a smooth k-sphere bundle has a smooth embedding in $G_{k+1}\mathbb{R}^{r+k+1}$, for r sufficiently large, such that B is transverse to the bad cone through each of its points.

PROOF: From THEOREM B we have produced a smooth great k-sphere fibration of a submanifold, $\tau\cdot(\psi\cdot\pi, f)(E)$ of S^N. Although we don't have a fibration of all of S^N by great k-spheres, the identical proof of ([G-W-Y], Theorem 4.1) carries through to produce the result. QED.

If P is a k-plane in $\mathbb{R}^n$, $k < n$, and $P^\perp$ denotes the orthogonal $(n-k)$-plane, and $\varphi : U \longrightarrow \mathrm{Hom}(P, P^\perp)$ is an embedding of an open subset U of a manifold X then since φ is injective, for $x \neq y$ in U, $\varphi(x) \neq \varphi(y)$. So for $x \neq y$ in U there exists $0 \neq v \in P$ with $\varphi(x)(v) \neq \varphi(y)(v)$.

DEFINITION 5.3. Given the situation just described, we say φ is a <u>strongly injective embedding</u> if for $x \neq y$ in U, $\varphi(x)(v) \neq \varphi(y)(v)$ for <u>all</u> $0 \neq v \in P$. φ is a <u>smooth strongly injective embedding</u> if, in addition, for all $0 \neq v \in P$, the map $\varphi_v : U \longrightarrow P^\perp$ given by $\varphi_v(x) = \varphi(x)(v)$ is an immersion.

If B is an embedded submanifold of $G_{k+1}\mathbb{R}^{N+1}$ such that B represents the base space of a great k-sphere fibration of some submanifold $E = (\bigcup_{Q \in B} Q) \cap S^N \subseteq S^N$, then for all $Q \in B$ and coordinate maps $\varphi_Q : U_Q \subseteq G_{k+1}\mathbb{R}^{N+1} \longrightarrow \mathrm{Hom}(Q, Q^\perp)$, $\varphi_Q | U_Q \cap B$ is a strongly injective embedding. In addition if B is the base space of a smooth great k-sphere fibration of E, then B is transverse to the bad cone through each of its points and $\varphi_Q | U_Q \cap B$ is a smooth strongly injective embedding.

Although these notions will be used more substantively in Part 3, we present here another interesting corollary to THEOREM B.

COROLLARY 5.4. Every compact C^{∞} n-dimensional manifold B has a smooth, strongly injective embedding in $Mat(k+1,(k+1)(2n+1))$, the $(k+1) \times (k+1)(2n+1)$ matricies over $\mathbb{R}$, for any $k \geq 0$.

PROOF: Apply the theorem to the trivial bundle $B \times S^k = E$. We may take the identity map for the f in THEOREM B, obtaining an embedding $(\phi \cdot \pi, id): E \longrightarrow S^{2n} \times S^k$. If P denotes the $(k+1)$-plane spanned by $e_1, \ldots, e_{k+1}$ in $\mathbb{R}^{N+1}$, where $N = (k+1)(2n+1) + k$, then note that the image of the map $\tau : S^{2n} \times S^k \longrightarrow S^N$ of the lemma is disjoint from $P^{\perp} \cap S^N$. Hence B embeds in $G_{k+1}\mathbb{R}^{N+1}$ and its image lies completely in the coordinate chart centered at P with coordinates given in $Hom(P, P^{\perp})$. QED.

We conclude Part 1 by briefly sketching the relation between strongly injective embeddings, regular algebra structures on $\mathbb{R}^n$, and fibrations of S^{2n-1} by great (n-1)-spheres. For a complete discussion of regular algebra structures on $\mathbb{R}^n$ and great n-sphere fibrations of S^{2n-1} see [YA-2] and [G-W-Y].

Every (smooth) great n-sphere fibration of S^{2n-1} with base space S^n gives a (smooth) strongly injective embedding $\varphi : \mathbb{R}^n \longrightarrow GL(n,\mathbb{R}) \cup \{0\}$ (where we view $\mathbb{R}^n$ as the base space S^n minus one point). Conversely, every (smooth) strongly injective embedding $\varphi : \mathbb{R}^n \longrightarrow GL(n,\mathbb{R}) \cup \{0\}$ (with $\varphi(0) = 0$) with a regularity condition on $\varphi(x)$ as $\|x\| \longrightarrow \infty$ (to assure differentiability at the fibre at ∞) gives a corresponding (smooth) great (n-1)-sphere fibration of S^{2n-1}.

If an embedding $\varphi : \mathbb{R}^n \longrightarrow GL(n,\mathbb{R}) \cup \{0\}$ is linear then it is easy to confirm that φ is a smooth strongly injective embedding. For suppose $x \neq y$ in $\mathbb{R}^n$. Then $\varphi(x-y) \in GL(n,\mathbb{R})$ so for $0 \neq v \in \mathbb{R}^n$, $0 \neq \varphi(x-y)(v) =$ $= \varphi(x)(v) - \varphi(y)(v)$ hence $\varphi(x)(v) \neq \varphi(y)(v)$. Such a φ gives a regular algebra structure to $\mathbb{R}^n$ (a bilinear multiplication with no zero divisors), $u : \mathbb{R}^n \times \mathbb{R}^n \longrightarrow \mathbb{R}^n$ via $u(a,b) = \varphi(a)(b)$. Also φ always gives a great (n-1)-sphere fibration of S^{2n-1}. Setting $P = \mathrm{span}(e_1,e_2,\ldots,e_{n+1}) \subseteq \mathbb{R}^{2n}$ we get the fibration by the graphs of the linear transformations $\varphi(a)$, $a \in \mathbb{R}^n$ viewed as maps from P to $P^{\perp}$, along with the fibre $P^{\perp}$.

Let $f : \mathbb{R}^{n^2} \longrightarrow \mathbb{R}$ be the polynomial $f(x_{11}, x_{12}, \ldots, x_{1n}, x_{21}, \ldots, x_{n,n-1}, x_{nn}) = \det([x_{ij}]_{i,j=1,\ldots,n})$ and set $V = f^{-1}(0)$. Note that $GL(n,\mathbb{R}) = \mathbb{R}^{n^2} - V$ and for any strongly injective embedding $\varphi : \mathbb{R}^n \longrightarrow \mathrm{Hom}(P,P^{\perp})$ (with P as above) with $\varphi(0) = 0$ we must have $\varphi(\mathbb{R}^n) \cap V = \{0\}$.

Using these notions we prove the main result in ([G-W-Y], Section 6). If U is any open neighborhood of 0 in $\mathbb{R}^n$ and $\varphi : U \longrightarrow GL(n,\mathbb{R}) \cup \{0\}$ is a smooth, strongly injective embedding with $\varphi(0) = 0$ then we get an associated linearization $d\varphi : \mathbb{R}^n \longrightarrow GL(n,\mathbb{R}) \cup \{0\}$ and hence an associated regular algebra structure on $\mathbb{R}^n$. To prove this we view the tangent space to $\mathbb{R}^{n^2}$ at the origin, $T_0\mathbb{R}^{n^2}$, as $\mathbb{R}^{n^2}$ itself. Since φ is an embedding we have $T_0\varphi(U)$ is an n-plane. Suppose $0 \neq v \in V$ and $v \in T_0\varphi(U)$. Let $b \in S^{n-1}$ be a vector such that $v(b) = 0$ ($v \in V$ implies v is a singular linear transformation since $\det(v) = f(v) = 0$) and let $\sigma : (-\epsilon,\epsilon) \longrightarrow \mathbb{R}^n$ be a smooth curve with $\sigma(0) = 0$ and $(\varphi\cdot\sigma)'(0) = v$ (so in particular $\sigma'(0) \neq 0$). Then

$$d\varphi_b(0)[\sigma'(0)] = \frac{d}{dt}\bigg|_{t=0} \varphi_b(\sigma(t)) = \frac{d}{dt}\bigg|_{t=0} \varphi(\sigma(t))(b) = v(b) = 0 ,$$

but this implies φ_b is not an immersion hence φ is not a smooth strongly injective embedding. This contradition allows us to conclude that the tangent space to $T_0\varphi(U)$ is a linear embedding of $\mathbb{R}^n$ in $GL(n,\mathbb{R})\cup\{0\}$.

PART 2.

Throughout this part, by abuse of notation, we let $S^m \times S^n$ denote the submanifold of S^{m+n+1} given by

$$S^m_{1/\sqrt{2}} \times S^n_{1/\sqrt{2}} = \{(x_1,\ldots,x_{m+n+2}) : \Sigma_{i=1}^{m+1} x_i{}^2 = \frac{1}{2} ,$$
$$\Sigma_{i=m+2}^{n+m+2} x_i{}^2 = \frac{1}{2}\}.$$

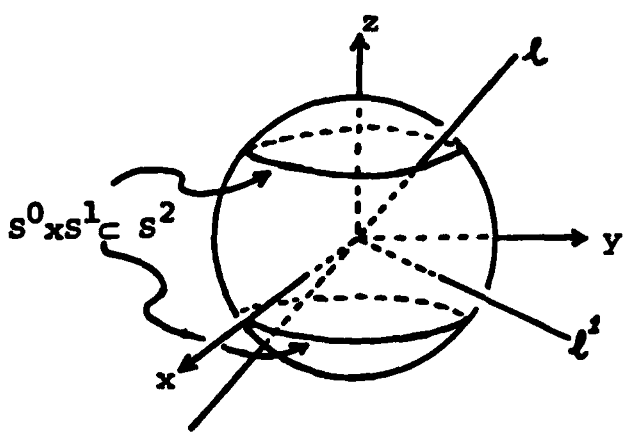
It should be clear from the context and thus cause no confusion when we write $S^m \times S^n$ whether we mean a product of unit spheres or a product of spheres of radius $1/\sqrt{2}$.

It should be remarked that we are examining a restricted situation. Trivially, $S^m \times S^n$ always admits a fibration by m-spheres and n-spheres, so by THEOREM B, $S^m \times S^n$ embeds in S^N for some N so that these fibres are great m-spheres or great n-spheres respectively. By analogy with the case of great 3-sphere fibrations of the

7-sphere mentioned in Section I, where we saw that there were many, topologically inequivalent such fibrations, but when we restricted the 7-sphere to S^7 all great 3-sphere fibrations were equivalent, we expect that $S^m \times S^n$ should admit a smaller class of <u>great</u> k-sphere fibrations.

Let $p_1 : \mathbb{R}^{m+n+2} \to \mathbb{R}^{m+1}$ be the projection on the first $m+1$ coordinates, and $p_2 : \mathbb{R}^{m+n+2} \to \mathbb{R}^{n+1}$ be the projection on the last $n+1$ coordinates.

LEMMA 5.5. A great k-sphere of S^{m+n+1} lying inside $S^m \times S^n$ gives an isometry from a great k-sphere in S^m onto a great k-sphere in S^n.



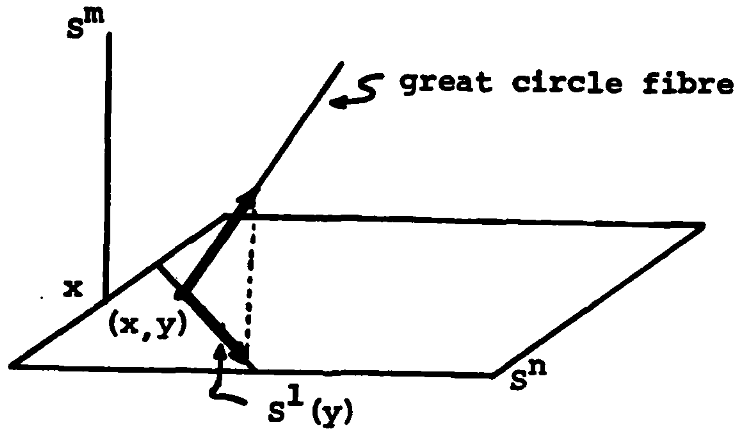
ℓ cuts out a great 0-sphere in S^2 contained in $S^1 \times S^0$. As such it represents the graph of an isometry from $\ell^1 = p_1(\ell)$ to $p_2(\ell)$ = z-axis.

PROOF: First we observe that $S^k \subseteq S^m \times S^n$ implies $k \leq \min(m,n)$. For suppose $m = \min(m,n)$ and $k > m$. Let P denote the $(k+1)$-plane spanned by S^k. $k+1+n+1 > m+n+2$ so P and $p_2(\mathbb{R}^{m+n+2}) = \mathbb{R}^{n+1}$ must intersect at least along a line hence there exists $v \in S^k \cap p_2(\mathbb{R}^{n+m+2})$. This v cannot be in $S^m \times S^n$ since $\|p_2(v)\| = \|v\| = 1$ but for all $w \in S^m \times S^n$, $\|p_2(w)\| = \|p_1(w)\| = 1/\sqrt{2}$. Hence we must have $k \leq \min(m,n)$.

This also shows that $P \cap p_1(\mathbb{R}^{m+n+2}) = P \cap p_2(\mathbb{R}^{m+n+2}) = \{0\}$ so the maps $p_1|P$ and $p_2|P$ both have rank $= k+1$ and kernel $= \{0\}$. Therefore $P_i = p_i(P)$ is a $(k+1)$-plane in $p_i(\mathbb{R}^{m+n+2})$ for $i = 1,2$ and P is the graph of a linear map, L_p, from P_1 to P_2.

For $v \in p_1(S^m \times S^n) \cap P_1$, $\|v\| = 1/\sqrt{2}$ and there is a $v_0 \in S^{m+n+1} \cap P$ with $p_1(v_0) = v$ and $v^1 = p_2(v_0) = L_p(v) \in p_2(S^m \times S^n)$. $\|v\| = 1/\sqrt{2} = \|v^1\| = \|L_p(v)\|$ hence L_p is an isometry. QED.

LEMMA 5.6. If m or n is even then $S^m \times S^n$ cannot be fibred by great circles of S^{m+n+1}.



PROOF: Say WLOG, n is even. Fix $x \in p_1(S^m \times S^n)$. If there were a fibration of $S^m \times S^n$ by great circles then for each $y \in p_2(S^m \times S^n)$, the great circle fibre through $(x,y) \in S^m \times S^n$ would project to a great circle $S^1(y)$ on $p_2(S^m \times S^n)$ through y. Each such circle $S^1(y)$ has a well defined tangent line at y, varying continuously with y on $p_2(S^m \times S^n) = S^n_{1/\sqrt{2}}$. But n is even so this is impossible. QED.

The previous two lemmas combine to give

COROLLARY 5.7. $S^1 \times S^{2n}$ has no great k-sphere fibrations for any n or $k \geq 1$.

So very quickly we see that, as we expected, restricting to <u>great</u> k-sphere fibrations of $S^m \times S^n$ does, in certain cases, considerably eliminate a number of topological k-sphere fibrations. Without too much more work we see that we can eliminate much more.

If n is even and $m \leq n$ then we can apply COROLLARY 5.7 to conclude that there is no great m-sphere fibration of $S^m \times S^n$. For suppose P is the $(n+3)$-dimensional plane in $\mathbb{R}^{m+n+2}$,

$$P = \{(x_1, x_2, \ldots, x_{m+n+2}): \ x_1 = x_2 = \ldots = x_{m-1} = 0\}.$$

Note that $P \cap S^{m+n+1} = S^{n+2}$ and $P \cap (S^n \times S^m) = S^1 \times S^n$. If there were a fibration of $S^m \times S^n$ by great m-spheres then since each fibre maps bijectively on $p_1(S^m \times S^n)$, intersecting the fibration with P would cut each fibre down to a great circle in S^{n+2} and give a great circle fibration of $S^1 \times S^n$. By COROLLARY 5.7 this is impossible for n even.

In particular we conclude $S^{2n} \times S^{2n}$ cannot be fibred by great $2n$-spheres.

What about great $(2n-1)$-sphere fibrations of $S^{2n-1} \times S^{2n-1}$? Clearly, any such fibration is trivial (let

$S_i^{2n-1} = p_i(S^{2n-1} \times S^{2n-1})$, then fix $q_0 \in S_2^{2n-1}$ and define $\psi : S_1^{2n-1} \times S^{2n-1} \longrightarrow S^{2n-1} \times S^{2n-1}$ by $\psi(r,q) = (r, L_{r,q_0}(q))$ where L_{r,q_0} is the isometry from S_1^{2n-1} to S_2^{2n-1} determined by the fibre through $(r,q_0))$. If S^{2n-1} denotes the base space of a great $(2n-1)$-sphere fibration of $S^{2n-1} \times S^{2n-1} \subseteq S^{4n-1} \subseteq \mathbb{R}^{4n}$ then from Part 1 we conclude that we have a strongly injective embedding $\varphi : S^{2n-1} \longrightarrow \mathrm{Hom}(p_1(\mathbb{R}^{4n}), p_2(\mathbb{R}^{4n}))$. Since every fibre in fact gives an isometry, if we identify the ith coordinate in $p_1(\mathbb{R}^{4n})$ with the $2n$ + ith coordinate of $p_2(\mathbb{R}^{4n})$ we can assume we have a *strongly injective embedding* $\varphi : S^{2n-1} \longrightarrow O(2n)$, the orthogonal group on $\mathbb{R}^{2n}$. Such a strongly injective embedding φ induces a map $\hat{\varphi} : S^{2n-1} \times S^{2n-1} \longrightarrow S^{2n-1}$, $\hat{\varphi}(a,b) = \varphi(a)(b)$. For any $b \in S^{2n-1}$, $\hat{\varphi}(\ ,b) : S^{2n-1} \longrightarrow S^{2n-1}$ and $\hat{\varphi}(b,\ ) : S^{2n-1} \longrightarrow S^{2n-1}$ are both injective so $\hat{\varphi}$ has bidegree $(1,1)$. By a theorem of Adams and Atiyah, ([HU], Chap 14) we conclude that $n = 1$, 3, or 7. Such strongly injective embeddings certainly exist in these dimensions, namely the unit spheres in $\mathbb{R}^2$, $\mathbb{R}^4$ and $\mathbb{R}^8$ considered as the complex numbers, quaternions, or Caley numbers respectively. Hence we have proven:

COROLLARY 5.8. $S^n \times S^n$ can be fibred by great n-spheres if and only if $n = 1$, 3 or 7.

Since S^{4n+1}, $n \geq 1$ does not admit a continuous field of tangent k-planes ([ST], Sect 27.18), $2 \leq k \leq 4n-1$, the idea in the proof of LEMMA 5.6 generalizes to

LEMMA 5.9. $S^m \times S^{4n+1}$ $n \geq 1$, admits no great k-sphere fibration for $2 \leq k \leq 4n-1$.

PROOF: To be specific, suppose we had a great 2-sphere fibration of $S^m \times S^{4n+1}$. Fix $x \in p_1(S^m \times S^{4n+1})$, then for each $y \in p_2(S^m \times S^{4n+1})$ let $T_{x,y}S^2$ be the tangent space to the great 2-sphere fibre at $(x,y) \in S^n \times S^{4n+1}$. Since the 2-sphere fibre projects onto an embedded great 2-sphere in $p_2(S^m \times S^{4n+1})$, $dp_2(T_{x,y}S^2)$ is a 2-plane $P_y \subseteq T_y p_2(S^m \times S^{4n+1})$. In this way we get a continuous field of tangent 2-planes on S^{4n+1}. But this is impossible.

QED.

Using these general facts we now turn our attention to some specific low dimensional cases.

1) Great sphere fibrations of $S^1 \times S^1 \subseteq S^3$.

Clearly the only possibility is for a fibration by great circles. From the discussion after Corollary 5.7, any such fibration is trivial with base space S^1 and it gives a strongly injective embedding of S^1 in $O(2)$. Now $O(2) = S^1 \underset{\text{disj}}{\cup} S^1$, so, modulo reparameterization, there are only two possible embeddings of S^1 in $O(2)$ with image either $SO(2)$ or $O(2) - SO(2)$. It is easy to see that either such embedding is a strongly injective embedding. So the space of great circle fibrations of $S^1 \times S^1$ is just 2 points, one point corresponding to a fibration by $(1,1)$ curves (homotopy type of typical fibre in $\pi_1(S^1 \times S^1)$) with typical fibre of the form

$$\{\frac{1}{\sqrt{2}}(e^{i\theta}, e^{i(\theta+\alpha)}) : \ 0 \leq \theta \leq 2\pi\},\ 0 \leq \alpha < 2\pi\ ,$$

and the other point corresponding to a fibration by $(1,-1)$ curves, with typical fibre

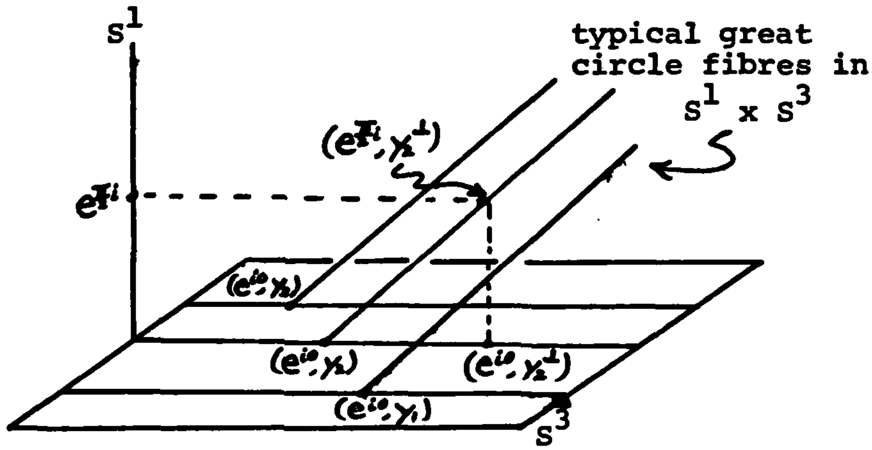
$$\{\frac{1}{\sqrt{2}}(e^{i\theta}, e^{i(-\theta+\alpha)}) : \ 0 \leq \theta \leq 2\pi\},\ 0 \leq \alpha < 2\pi\ .$$

2) By COROLLARY 5.7, $S^1 \times S^2$ admits no great sphere fibrations.

3) By LEMMA 5.6 and the discussion after COROLLARY 5.7,

$S^2 \times S^2$ admits no great sphere fibrations.

4) Great circle fibrations of $S^1 \times S^3 \subseteq S^5$.



Given a great circle fibration of S^3 we get a great circle fibration of $S^1 \times S^3$ as illustrated in the picture above. Any fibration of S^3 is orientable so assume it oriented and for $y \in S^3$, let $y^{\perp}$ denote that element of S^3 gotten by rotating $\pi/2$ radians in the oriented direction along the fibre through y . For each $y \in S^3$,

$$S(y) = \{\frac{1}{\sqrt{2}} \cos\theta(1,0,q) + \frac{1}{\sqrt{2}} \sin\theta(0,1,q^{\perp}) : \ 0 \le \theta \le 2\pi\}$$

is a great circle of S^5 lying entirely in $S^1 \times S^3$. Its easy to see that the family of all such great circles fibres

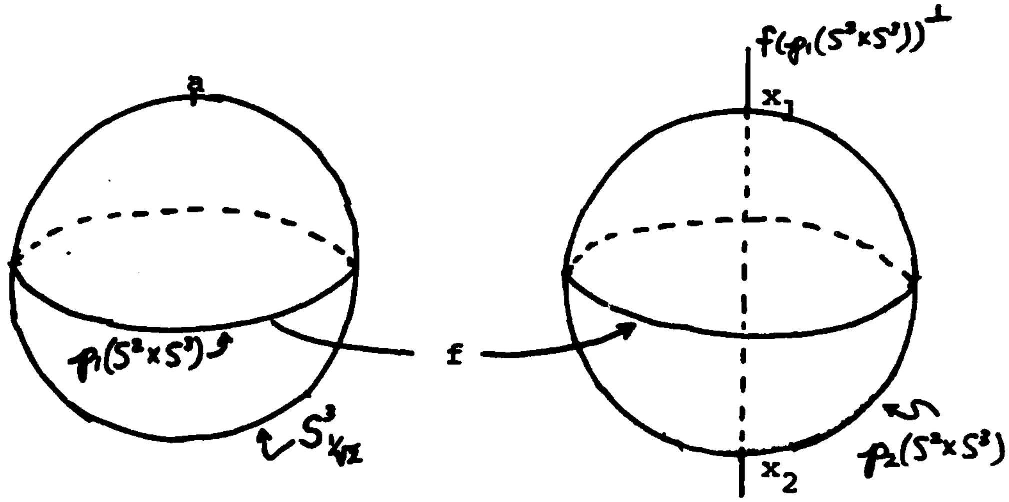
$S^1 \times S^3$.

As a consequence of our work in Part 3 below we will see that a great circle fibration can be obtained in a natural way from any distance decreasing map from S^3 to S^2 . Certain of these maps will give fibrations of $S^1 \times S^3$ that do not correspond to great circle fibrations of the S^3 factor as above. It is not clear whether every great circle fibration of $S^1 \times S^3$ arises from a distance decreasing map of S^3 to S^2 so this remains a basic open question.

5) Great sphere fibrations of $S^2 \times S^3$

The only possibility is for a fibration by great 2-spheres. In Part 3 below we completely catalogue all great 3-sphere fibrations of $S^3 \times S^3$. Suppose we have a great 3-sphere fibration of $S^3 \times S^3 \subseteq S^7$. Let $S^6 \subseteq S^7$ be given by $S^6 = S^7 \cap e_1{}^{\perp}$. Note that $S^2 \times S^3 = (S^3 \times S^3) \cap e_1{}^{\perp}$. Exactly as in the discussion following COROLLARY 5.7 we conclude that any great 3-sphere fibration of $S^3 \times S^3$, desuspends, via intersection with $e_1{}^{\perp}$ to a great 2-sphere fibration of $S^2 \times S^3$.

Conversely, we now show that any great 2-sphere fibration of $S^2 \times S^3$ can be extended in one of 2 distinct ways to a great 3-sphere fibration of $S^3 \times S^3$. By LEMMA 5.5, a great 2-sphere fibre gives an isometry from the first factor, $p_1(S^2 \times S^3) = S^2_{1/\sqrt{2}}$ to a great 2-sphere in $p_2(S^2 \times S^3) = S^3_{1/\sqrt{2}}$. If we view the first factor as sitting inside $S^3_{1/\sqrt{2}}$ then the isometry from $S^2_{1/\sqrt{2}}$ into $S^3_{1/\sqrt{2}}$ extends in only one of two ways to an isometry from $S^3_{1/\sqrt{2}}$ to $S^3_{1/\sqrt{2}}$.



f has two suspensions to an isometry from $S^3_{1/\sqrt{2}}$ to $p_2(S^2 \times S^3) = S^3_{1/\sqrt{2}}$ determined by $a \longrightarrow x_1$ or $a \longrightarrow x_2$.

So suppose given a great 2-sphere fibration of $S^2 \times S^3 \subseteq S^6 \subseteq \mathbb{R}^7$ where $\mathbb{R}^7 \subseteq \mathbb{R}^8$ as $e_1^{\perp}$ (all first coordinates zero).

$$S^2 \longrightarrow S^2 \times S^3 \xrightarrow{pr} B$$

Let $a_2 = (0,\sqrt{2}/2,0,0)$, $a_3 = (0,0,\sqrt{2}/2,0)$, $a_4 = (0,0,0,\sqrt{2}/2)$ be 3 points in $p_1(S^2 \times S^3)$. For each $x \in B$

$$pr^{-1}(x) = \{\Sigma_{i=2}^{4} \alpha_i(a_i,x_i) : \Sigma_{i=2}^{4} \alpha_i^{\,2} = 1 ,$$
$$x_2,x_3,x_4 \in p_2(S^2 \times S^3) \text{ with}$$
$$\frac{2}{\sqrt{2}}(x_2,x_3,x_4) \text{ an orthonormal}$$
$$\text{3-frame in } \mathbb{R}^4\}$$

So the correspondence $x \longrightarrow \frac{2}{\sqrt{2}}(x_2,x_3,x_4)$ embeds B in $V_3\mathbb{R}^4$ and any such fibration of $S^2 \times S^3$ is trivial with $B \simeq S^3$.

Now we have a fibration $S^0 \longrightarrow V_4\mathbb{R}^4 \xrightarrow{q} V_3\mathbb{R}^4$ where q applied to any 4-frame simply ignores the first vector of the frame. This is just the double covering, $V_4\mathbb{R}^4 = O(4) \simeq SO(4) \underset{\text{disj}}{\cup} SO(4)$ and $V_3\mathbb{R}^4 \simeq SO(4)$. So the base space B of our great 2-sphere fibration of $S^2 \times S^3$ has 2 distinct (otherwise unique) lifts to $V_4\mathbb{R}^4$. Suppose we lift B to $SO(4) \subseteq V_4\mathbb{R}^4$, $p : B \longrightarrow SO(4)$. Let $a_1 = (\sqrt{2}/2,0,0,0)$, and $q_i : B \longrightarrow S^3$, $1 \leq i \leq 4$ the map gotten by projecting the 4×4 matrix $p(x)$, $x \in B$, onto its first, second, third, or fourth column vector respectively. For each $x \in B$ we get a great 3-sphere in $S^7 \subseteq \mathbb{R}^8$, lying entirely in $S^3 \times S^3$:

$$x \longrightarrow \{\Sigma_{i=1}^{4} \alpha_i (a_i, \frac{1}{\sqrt{2}} q_i(x)) : \Sigma_{i=1}^{4} \alpha_i^2 = 1\}$$

Suppose two such great 3-spheres intersected. This means

$$\frac{1}{\sqrt{2}}((\alpha_1,\alpha_2,\alpha_3,\alpha_4),\Sigma_{i=1}^{4} \alpha_i q_i(x)) = \frac{1}{\sqrt{2}}((\beta_1,\beta_2,\beta_3,\beta_4),\Sigma_{i=1}^{4} \beta_i q_i(y))$$

where $\Sigma_{i=1}^{4} \alpha_i^2 = \Sigma_{i=1}^{4} \beta_i^2 = 1$, and $x \neq y$ are both in B. Immediately we see $\alpha_i = \beta_i$, $1 \leq i \leq 4$, and if $\alpha = (\alpha_1,\alpha_2,\alpha_3,\alpha_4)$, then $p(x)(\alpha) = p(y)(\alpha)$, where $p(x)$ and $p(y)$ are elements of $SO(4)$. But if two elements of $SO(4)$ agree at one non-zero vector, they in fact agree along an entire 2-plane in $\mathbb{R}^4$ ($A_1 v = A_2 v$, A_1, A_2 in $SO(4)$ implies $A_2^{-1} A_1 v = v$ so $A_2^{-1} A_1 : v^{\perp} \longrightarrow v^{\perp}$ and $A_2^{-1} A_1 | v^{\perp} \in SO(3)$ hence $\exists\, 0 \neq w \in v^{\perp}$ with $A_2^{-1} A_1 w = w$). So there must be a vector of the form $(0,\gamma_2,\gamma_3,\gamma_4) = \gamma \in S^3$ with $p(x)\gamma = p(y)\gamma$. Hence

$$\frac{1}{\sqrt{2}}((0,\gamma_2,\gamma_3,\gamma_4),\Sigma_{i=1}^{4} \gamma_i q_i(x)) = \frac{1}{\sqrt{2}}((0,\gamma_2,\gamma_3,\gamma_4),\Sigma_{i=2}^{4} \gamma_i q_i(y))$$

and $pr^{-1}(x) \cap pr^{-1}(y) \neq \emptyset$. This contradicts the fact that pr is a projection of a fibration and $x \neq y$. Therefore no two such 3-spheres can intersect and the family of all such great 3-spheres fibres $S^3 \times S^3$. Clearly if we apply the process of intersecting this fibration with $e_1^{\perp}$ we recover our original great 2-sphere fibration of $S^2 \times S^3$ hence these two operations are inverse.

Had we lifted B to $O(4) - SO(4)$ we would have obtained another fibration of $S^3 \times S^3$. So in summary we have the CONCLUSION: There is a 2-to-1 correspondence between great 3-sphere fibrations of $S^3 \times S^3$ and great 2-sphere fibrations of $S^2 \times S^3$. Therefore all the results we obtain in Part 3 pertaining to great 3-sphere fibrations of $S^3 \times S^3$ can, with minor modification, be applied to great 2-sphere fibrations of $S^2 \times S^3$.

6) Great circle fibrations of $S^3 \times S^3 \subseteq S^7$

Let

$$F_1: \; S^1 \longrightarrow S^3_{1/\sqrt{2}} \qquad F_2: \; S^1 \longrightarrow S^3_{1/\sqrt{2}}$$
$$\begin{array}{ccc} \downarrow q_1 & \qquad\qquad & \downarrow q_2 \\ S^2 & & S^2 \end{array}$$

be any two oriented great circle fibrations of $p_i(S^3 \times S^3)$, $i = 1$ and 2 respectively. Given a great circle fibre on the first factor and one on the second factor their product is an $S^1 \times S^1 \subseteq S^3 \times S^3 \subseteq S^7$. By (1) this $S^1 \times S^1$ admits a unique fibration by $(1,1)$ great circles (since the fibrations are oriented the notion of $(1,1)$ makes sense on all such $S^1 \times S^1 \subseteq S^3 \times S^3$). In this way we can associate to any pair F_1 and F_2 a great circle fibration of $S^3 \times S^3$.

As in paragraph (4) above there remain unanswered questions here also. Are all great circle fibrations of $S^3 \times S^3$ obtained by a product of two such fibrations of the factors?

7) Great 2-sphere fibrations of $S^3 \times S^3$.

None of the above lemmas address the case of fibrations of $S^3 \times S^3$ by great 2-spheres. From the Gysin sequence we can settle this case by proving that in fact $S^3 \times S^3$ does not even admit a topological fibration by 2-spheres.

Suppose we had a fibration, $S^2 \longrightarrow S^3 \times S^3 \longrightarrow M^4$. Since $S^3 \times S^3$ is simply connected and S^2 is path connected M^4 must be connected and simply connected, hence

$H^0(M,\mathbb{Z}) = \mathbb{Z}$ and the fibration is orientable. The Gysin sequence of our hypothetical fibration gives:

$$\begin{array}{ccccccc} \to H^5(M^4) & \to & H^5(S^3 \times S^3) & \to & H^3(M^4) & \to & H^6(M^4) \\ \| & & \| & & & & \| \\ 0 & & 0 & & & & 0 \end{array}$$

So by exactness we must have $H^3(M^4) = 0$. But another segment of the Gysin sequence gives:

$$\begin{array}{ccccccc} \to H^2(M^4) & \to & H^2(S^3 \times S^3) & \to & H^0(M^4) & \to & H^3(M^4) \to \\ & & \| & & | & & \\ & & 0 & & \mathbb{Z} & & \end{array}$$

So the conclusion $H^3(M^4) = 0$ destroys the exactness of this segment. Hence such a fibration cannot fit into a Gysin sequence so it must not exist. Therefore $S^3 \times S^3$ admits no great 2-sphere fibration.

8) Examples of $S^m \times S^n$ which admit no great k-sphere fibrations, $k > 0$.

$S^1 \times S^2$	$S^2 \times S^2$	$S^2 \times S^5$	$S^4 \times S^9$
$S^1 \times S^4$	$S^2 \times S^4$	$S^2 \times S^9$	$S^4 \times S^{13}$
$S^1 \times S^6$	$S^2 \times S^6$	$S^2 \times S^{13}$	$S^6 \times S^9$

$S^1 \times S^8 \quad S^2 \times S^8 \quad S^2 \times S^{17} \quad S^6 \times S^{13}$

PART 3.

In this part we examine in detail the case of great 3-sphere fibrations of $S^3 \times S^3 \subseteq S^7$. From Part 2, we know that such a fibration is trivial with base space S^3 and it gives a strongly injective embedding of S^3 in $O(4)$. Conversely, suppose $\varphi : S^3 \longrightarrow O(4)$ is a strongly injective embedding. For $x \in S^3$, the set $\{\frac{1}{\sqrt{2}}(b, \varphi(x)(b)) : b \in S^3\}$ is clearly a great 3-sphere that lies in $S^3 \times S^3$. Since the embedding is strongly injective, for fixed $b \in S^3$, the map from S^3 to S^3, $x \longrightarrow \varphi(x)(b)$, is injective hence a homeomorphism. So for $(b,a) \in S^3 \times S^3$ there is an $x \in S^3$ such that $\varphi(x)(b) = a$. Also φ is a strongly injective embedding implies, for $x \neq y$, $(\mathrm{Graph}\ \varphi(x) \cap S^7) \cap (\mathrm{Graph}\ \varphi(y) \cap S^7) = \emptyset$. Therefore we conclude: There is a bijective correspondence between (smooth) great 3-sphere fibrations of $S^3 \times S^3$ and the image in $O(4)$ of (smooth) strongly injective embeddings $\varphi : S^3 \longrightarrow O(4)$.

So our approach to studying great 3-sphere fibrations of $S^3 \times S^3$ will be to analyze the equivalent problem of strongly injective embeddings of S^3 in $O(4)$. Since $O(4)$ has two connected components, we lose no generality by restricting our study to strongly injective embeddings of S^3 in $SO(4)$. It is well known that $SO(4) \simeq S^3 \times \mathbb{R}P^3$ hence $SO(4)$ has $S^3 \times S^3$ as double cover. In all that follows we identify $\mathbb{R}^4$ with the quaternions in the usual manner and S^3 will represent the quaternions of norm one. With these identifications we get the double cover projection $h : S^3 \times S^3 \longrightarrow SO(4)$

$$h(u,v)(x) = uxv \quad \text{(quaternion multiplication)}$$

Suppose $\tilde{\varphi} : S^3 \longrightarrow SO(4)$ is a strongly injective embedding with $\tilde{\varphi}_1 : S^3 \longrightarrow S^3$ given by $\tilde{\varphi}_1(v) = \tilde{\varphi}(v)(1)$. Since $\tilde{\varphi}$ is a strongly injective embedding, $\tilde{\varphi}_1$ is injective hence it is a homeomorphism. Let $\varphi = \tilde{\varphi}\cdot\tilde{\varphi}_1^{-1} : S^3 \longrightarrow SO(4)$. If $w = \tilde{\varphi}_1^{-1}(v)$ then $\tilde{\varphi}(w)(1) = v$ and $\varphi_1(v) = \varphi(v)(1) = [\tilde{\varphi}\cdot\tilde{\varphi}_1^{-1}(v)](1) = \tilde{\varphi}(w)(1) = v$. Now φ is just a reparameterization of φ hence Image φ = Image $\tilde{\varphi}$ and φ is a strongly injective embedding which determines exactly the same great 3-sphere fibration of $S^3 \times S^3$ as $\tilde{\varphi}$. So given a strongly injective embedding $\varphi : S^3 \longrightarrow SO(4)$ we may and shall assume that $\varphi_1(v) = v$ for all $v \in S^3$.

We have

$$\begin{tikzcd} & S^3 \times S^3 \arrow[d, "h\ \text{(double cover)}"] \\ S^3 \arrow[ur, dashed, "\hat{\varphi}"] \arrow[r, "\varphi"'] & SO(4) \end{tikzcd}$$

Since S^3 is simply connected φ lifts to a map $\hat{\varphi} : S^3 \longrightarrow S^3 \times S^3$ such that $h \cdot \hat{\varphi} = \varphi$. $\hat{\varphi}$ is unique up to choice of base point lying over say $\varphi(1)$. So $\hat{\varphi}(v) = (f(v), g(v)) \in S^3 \times S^3$ and

$$v = \varphi_1(v) = \varphi(v)(1) = h(f(v), g(v))(1) = f(v)g(v)$$

Therefore $g(v) = [f(v)]^{-1}v$ and g depends uniquely on f which depends, modulo choice of $h^{-1}\varphi(1)$, on the strongly injective embedding φ . So strongly injective embeddings depend only on a single map $f : S^3 \longrightarrow S^3$.

Now we address the question: What criteria are there to guarantee that a 3-sphere embedded in $S^3 \times S^3$ projects via h to the image of a strongly injective embedding?

Let $d : S^3 \times S^3 \longrightarrow [0, \pi]$ denote distance on the unit sphere S^3, $d(x,y)$ = least value of the length of the great circle ar joining x and y. Since multiplication by a norm 1 quaternion is an isometry of S^3, we get immediately,

$$d\ x,y) = d(xq,yq) = d(qx,qy) \quad \text{for all} \quad x,y,q \in S^3$$

LEMMA 5.11. A necessary condition for $h(x,y)(w) = h(u,v)(w)$ for some $w \in S^3$ is that $d(x,u) = d(y,v)$.

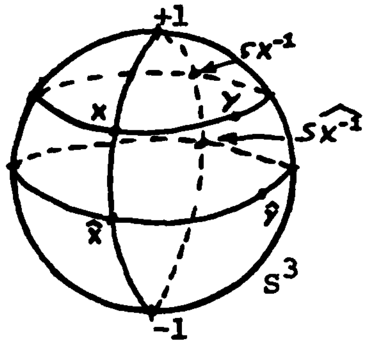
PROOF: $xwy = uwv \Leftrightarrow xwyv^{-1} = uw$, so $d(x,u) = d(xw,uw) =$ $= d(xw,xwyv^{-1}) = d(1,yv^{-1}) = d(v,y) = d(y,v)$. QED.

LEMMA 5.12. $h(x,y) \in SO(4)$ has a fixed point (+1 eigenvalue) if and only if $d(x,1) = d(y,1)$.

PROOF: ($\Rightarrow$) $h(x,y)(w) = w = h(1,1)(w)$ so by LEMMA 5.11, $d(x,1) = d(y,1)$.

($\Leftarrow$) If $x = \pm 1$ then $d(x,1) = d(y,1)$ implies $y = \pm 1$ and both $h(1,1)$ and $h(-1,-1)$ have fixed points, so the conclusion is true for $x = \pm 1$.

For $x \neq \pm 1$, suppose $d(x,1) = d(y,1) = c$ with $0 < c < \pi$. Let $\hat{x}$ denote the quaternion on the great circle through 1 and x with $d(1,\hat{x}) = \pi/2$ ($\hat{x} \in 1^{\perp}$) and $d(x,\hat{x}) < \pi/2$. We recall that $SO(3) \approx \mathbb{R}P^3$ and one way to make this identification is to conjugate by norm 1



quaternions and this action restricted to the unit *2-sphere* in S^3 of quaternions with real part 0. So there exists $w \in S^3$ such that $w^{-1}\widehat{x^{-1}}w = \hat{y}$.

Now conjugation by w is an isometry of S^3 fixing ± 1 so it takes the great circle through 1 and $\widehat{x^{-1}}$ to the great circle through 1 and $\hat{y}$.

But $d(1,w^{-1}x^{-1}w) = d(w,x^{-1}w) = d(1,x^{-1}) = d(x,1) =$ $= d(y,1)$ and $\pi/2 > d(x^{-1},\widehat{x^{-1}}) = d(w^{-1}x^{-1}w,w^{-1}\widehat{x^{-1}}w) =$ $= d(w^{-1}x^{-1}w,\hat{y})$ together imply we must have $y = w^{-1}x^{-1}w$. This implies $w = xwy = h(x,y)(w)$ and $h(x,y)$ has a fixed point. QED.

THEOREM 5.13. Given $x,y,u,v \in S^3$, $h(x,y)(w) = h(u,v)(w)$ for some $w \in S^3$, if and only if, $d(x,u) = d(y,v)$.

PROOF: ($\Rightarrow$) LEMMA 5.11.

($\Leftarrow$) $d(x,y) = d(y,v)$ implies $d(1,x^{-1}u) = d(x,u) =$ $= d(y,v) = d(1,vy^{-1})$. So by LEMMA 5.12, there exists $w \in S^3$ with $w = h(x^{-1}u,vy^{-1})(w) = x^{-1}uwvy^{-1}$ hence $h(x,y)(w) = xwy = uwv = h(u,v)(w)$. QED.

This is the key result we were after and we will see shortly that it will allow us to completely characterize strongly injective embeddings of S^3 in $SO(4)$. Before we do that, however, we digress momentarily to reconsider great circle fibrations of $S^1 \times S^3$ and settle a claim made in Part 2, paragraph 4.

Recall that we saw in Part 2, paragraph 4 that every great circle fibration of S^3 gave rise to a great circle fibration of $S^1 \times S^3$ by "lifting" each fibre so that as we went around the great circle in S^3 we also went around the first factor, S^1. Projecting such a fibration onto the second factor recovers the original great circle fibration of the second factor. The question asked was whether every great circle fibration of $S^1 \times S^3$ projects to a great circle fibration of S^3. As in LEMMA 5.6, if we fix $p = (\frac{\sqrt{2}}{2}, 0)$ on the S^1 factor, every great circle fibration of $S^1 \times S^3$ gives us a vector field on the S^3 factor. For each $q \in S^3$, the fibre (which we assume oriented) projects to a great circle through q on S^3 and we take the unit tangent vector to this circle at q, call it $F(q)$. Clearly, $F(q)$ is just the projection on S^3 of the point on the fibre through (p,q) whose projection on S^1 is $(0, \frac{\sqrt{2}}{2})$. Conversely, given a unit tangent vector field $F : S^3 \to S^3$,

$F(q) \in q^{\perp}$, we would try to fibre $S^1 \times S^3$ by taking the collection of great circles on S^5:

$$S^1(q) = \{\cos\theta(\tfrac{\sqrt{2}}{2},0,q)+\sin\theta(0,\tfrac{\sqrt{2}}{2},F(q)): \ 0 \leq \theta \leq 2\pi\}$$

for all $q \in p_2(S^1 \times S^3)$. Of course for arbitrary F this collection will not fibre $S^1 \times S^3$. As a minimum F must be injective for if $F(q_1) = F(q_2)$ then $(0,\frac{\sqrt{2}}{2},F(q_1)) \in S^1(q_1) \cap S^1(q_2)$ and two "fibres" would intersect. If F is injective then no two such great circles in $S^1 \times S^3$ will intersect at $\theta = 0$ and 2π and $\theta = \frac{\pi}{2}$ and $\frac{3\pi}{2}$. Since two or more such great circles intersecting is the only thing that would prohibit this family from fibring $S^1 \times S^3$ we conclude:

If $F: S^3 \to S^3$, $F(q) \in q^{\perp}$ for $q \in S^3$, and if we define $H: [0,2\pi] \times S^3 \to S^3$ by $H(q,\theta) = \cos\theta \cdot q + \sin\theta \cdot F(q)$ then the family of great circles in S^5:

$$S^1(q) = \{\tfrac{\sqrt{2}}{2}(\cos\theta,\sin\theta,H(q,\theta)): \ 0 \leq \theta \leq 2\pi\} \text{ for all } q \in S^3,$$

fibres $S^1 \times S^3$ if and only if $H(\ ,\theta): S^3 \to S^3$ is injective for all $0 \leq \theta \leq 2\pi$.

AD-A132 544 GREAT SPHERE FIBRATIONS OF MANIFOLDS(U) AIR FORCE INST OF TECH WRIGHT-PATTERSON AFB OH J PETRO 1983 AFIT/CI/NR-83-27D

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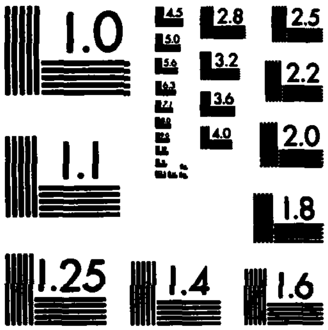

MICROCOPY RESOLUTION TEST CHART
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For $q \in S^3$, the imag of q under H as θ goes from 0 to 2π is clearly a great circle of S^3 and this great circle is just the projection of $S^1(q)$ on the second factor. So we are seeking a map H as above such that $H(\ ,\theta)$ is injective for all θ yet the flow of H does not determine a great circle fibration of S^3.

First note that there is a bijective correspondence between smooth, unit vector fields on S^3 and smooth maps $S^3 \longrightarrow S^2$ where $S^2 \subseteq S^3$ and S^2 represents the purely imaginary norm 1 quaternions. Given F as above, define $f : S^3 \longrightarrow S^2$ by $f(p) = p^{-1}F(p)$ (quaternion multiplication). Note that $\frac{\pi}{2} = d(p,F(p)) = d(1,p^{-1}F(p))$ hence $\operatorname{Im} f \subseteq S^2$. Conversely given $f : S^3 \longrightarrow S^2$, let $F : S^3 \longrightarrow S^3$ be given by $F(p) = pf(p)$. Its clear that $F(p) \in p^{\perp}$.

In terms of f the map H above is given by

$$H(p,\theta) = \cos\theta\cdot p + \sin\theta\, pf(p) = p(\cos\theta + \sin\theta\, f(p)).$$

For each p and θ we get an element of $SO(4)$:

$$(p,\theta) \longrightarrow h(p,\cos\theta + \sin\theta\cdot f(p)) \in SO(4).$$

For brevity we denote this element by $h(p,\theta)$.

If H is not injective for some value θ_0 of θ then $H(p,\theta_0) = H(q,\theta_0)$ for $p \neq q$ and consequently

$$h(p,\theta_0)(1) = H(p,\theta_0) = H(q,\theta_0) = h(q,\theta_0)(1). \qquad (*)$$

So by THEOREM 5.13 we conclude

$$d(p,q) = d(\cos\theta_0 + \sin\theta_0 f(p), \cos\theta_0 + \sin\theta_0 f(q)).$$

Now as θ goes from 0 to 2π, $\cos\theta + \sin\theta\, f(q)$ goes around the great circle through 1 and $f(q)$, hence

$$d(\cos\theta + \sin\theta \cdot f(p), \cos\theta + \sin\theta \cdot f(q)) \leq d(f(p), f(q)) \quad \forall\theta$$

and if f is distance decreasing, then for all $p \neq q$ and all θ

$$d(p,q) > d(f(p),f(q)) \geq d(\cos\theta + \sin\theta\, f(p), \cos\theta + \sin\theta\, f(q)).$$

Therefore, again by THEOREM 5.13,

$$H(p,\theta) = h(p,\theta)(1) \neq h(q,\theta)(1) = H(q,\theta)$$

and $H(\ ,\theta)\colon S^3 \longrightarrow S^3$ is injective for all values of θ.

If $f : S^3 \longrightarrow \{i\} \subseteq S^2$ is the constant map then one can easily check that the flow along the related map H determines a great circle fibration of S^3 (the Hopf fibration). Now let $\tilde{f}$ be a perturbation of f in a neighborhood of 1 such that it is still distance decreasing and $\tilde{f}(x) = i$ for all x with $d(x,1) \geq \pi/4$ and $\tilde{f}(1) \neq i$ Since $\tilde{f}$ is distance decreasing $\tilde{H}(\ ,\theta)\colon S^3 \longrightarrow S^3$ is still injective for all θ, however, if

$$\tilde{S}(1) = \{\tilde{H}(1,\theta)\colon\ 0 \leq \theta \leq 2\pi\},\ \tilde{S}(i) = \{\tilde{H}(i,\theta)\colon\ 0 \leq \theta \leq 2\pi\}$$

then $1 \in \tilde{S}(1) \cap \tilde{S}(i)$ but $\tilde{S}(1) \neq \tilde{S}(i)$ since $i \in \tilde{S}(i)$ but $i \notin \tilde{S}(1)$. Therefore $\tilde{H}$ does not determine a great circle fibration of S^3. But for each $q \in S^3$

$$S(q) = \{\frac{1}{\sqrt{2}}(\cos\theta, \sin\theta, \tilde{H}(q,\theta))\colon\ 0 \leq \theta \leq 2\pi\}$$

is a great circle in $S^1 \times S^3 \subseteq S^5$ and $\tilde{H}(\ ,\theta)$ injective implies the family of all such great circles fibres $S^1 \times S^3$.

So for every distance decreasing map $f\colon S^3 \longrightarrow S^2$ we get a great circle fibration of $S^1 \times S^3$. Unfortunately, in equation (*) above we are only applying the element $h(p,\theta) \in SO(4)$ to the vector $1 \in S^3$ so we cannot apply the converse of THEOREM 5.13 to conclude that $H(\ ,\theta)$ injective for all θ implies the related map $f\colon S^3 \longrightarrow S^2$ is distance decreasing. Whether this is so or not remains an interesting, open question.

Now we turn our attention back to using THEOREM 5.13 to study strongly injective embeddings of S^3 in $SO(4)$.

THEOREM 5.14. A submanifold of $S^3 \times S^3$ corresponds to the image of a strongly injective embedding of S^3 in $SO(4)$, if and only if, it is the graph of a smooth distance decreasing map ϕ from either S^3 factor to the other.

PROOF: Using THEOREM 5.13, the proof is identical to ([G-W], Theorem A).

THEOREM 5.15: $\varphi : S^3 \longrightarrow SO(4)$ is a smooth strongly injective embedding if and only if the corresponding distance decreasing map ϕ is differentiable with $|d\phi| < 1$.

PROOF: ($\Rightarrow$) By THEOREM 5.14, φ corresponds to a smooth distance decreasing map ϕ from one factor of S^3 to the other. Therefore we have $|d\phi| \leq 1$.

Suppose $d\phi = 1$ at some point $(u,v) \in S^3 \times S^3$, $v = \phi(u)$. Left and right multiplication in the Lie group S^3 are both diffeomorphisms of norm 1, so replacing h by $h(x,y) = h(u^{-1}x, yv^{-1})$ we may and shall assume $(u,v) = (1,1)$.

$|d\phi| = 1$ implies there is a parameterized curve $\sigma : (-\varepsilon,\varepsilon) \longrightarrow S^3$, $\sigma(0) = 1$, $\sigma'(0) = V$ with $|V| = 1$

such that $|(\psi\cdot\sigma)'(0)| = |V| = 1$. We assume σ traverses a portion of a great circle, and by a conjugation action applied to the first factor which rotates the purely imaginary 2-sphere we can suppose $\sigma(t) = \cos t + i \sin t$.

Now $\psi\cdot\sigma(t) = \psi_1(t) + \psi_2(t)i + \psi_3(t)j + \psi_4(t)k$, with $\psi_1(0) = 1$, $\psi_i(0) = 0 \quad 2 \le i \le 4$; $\psi_1'(0) = 0$, $\psi_2'(0)^2 + {}$ $+ \psi_3(0)^2 + \psi_4'(0)^2 = 1$.

The matrix for $h(\sigma(t), \psi\cdot\sigma(t)) \in SO(4)$ is given by:

$$H(t) = \begin{bmatrix} \cos t & -\sin t & 0 & 0 \\ \sin t & \cos t & 0 & 0 \\ 0 & 0 & \cos t & -\sin t \\ 0 & 0 & \sin t & \cos t \end{bmatrix} \begin{bmatrix} \psi_1(t) & -\psi_2(t) & -\psi_3(t) & -\psi_4(t) \\ \psi_2(t) & \psi_1(t) & \psi_4(t) & -\psi_3(t) \\ \psi_3(t) & -\psi_4(t) & \psi_1(t) & \psi_2(t) \\ \psi_4(t) & \psi_3(t) & \psi_2(t) & \psi_1(t) \end{bmatrix}$$

$$\left.\frac{d}{dt}\right|_{t=0} H(t) = \begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \cdot I$$

$$+ I \cdot \begin{bmatrix} 0 & -\psi_2'(0) & -\psi_3'(0) & -\psi_4'(0) \\ \psi_2'(0) & 0 & \psi_4'(0) & -\psi_3'(0) \\ \psi_3'(0) & -\psi_4'(0) & 0 & \psi_2'(0) \\ \psi_4'(0) & \psi_3'(0) & -\psi_2'(0) & 0 \end{bmatrix}$$

$$= \begin{bmatrix} & -1-\psi_2'(0) & -\psi_3'(0) & -\psi_4'(0) \\ 1+\psi_2'(0) & 0 & \psi_4'(0) & -\psi_3'(0) \\ \psi_3'(0) & -\psi_4'(0) & 0 & -1+\psi_2'(0) \\ \psi_4'(0) & \psi_3'(0) & 1-\psi_2'(0) & 0 \end{bmatrix}$$

It is a straightforward but tedious calculation to compute:

$$\det[\frac{d}{dt}\Big|_{t=0} H(t)] = (-1+\psi_2'(0)^2+\psi_3'(0)^2+\psi_4'(0)^2)^2 \quad (5.15.1)$$

So $|(\psi\circ\sigma)'(0)| = |\sigma'(0)| = 1$ implies $\det[\frac{d}{dt}\Big|_{t=0} H(t)] = 0$.

Let $w \in S^3$ be a vector such that

$$[\frac{d}{dt}\Big|_{t=0} H(t)](w) = 0$$

Now we suppose $\varphi : S^3 \longrightarrow S^3 \times S^3$ is given $\varphi = (id, \psi)$.

Then we get

$$(d\varphi_w)_1(V) = \frac{d}{dt}\Big|_{t=0} \varphi_w\circ\sigma(t) = \frac{d}{dt}\Big|_{t=0} h(\sigma(t), \psi\circ\sigma(t))(w) =$$

$$= [\frac{d}{dt}\Big|_{t=0} H(t)](w) = 0 \quad (5.15.2)$$

hence φ_w is not a diffeomorphism at $1 \in S^3$. Therefore φ is not a smooth strongly injective embedding.

($\Leftarrow$) We already know that given a distance decreasing map $\psi : S^3 \longrightarrow S^3$ we get a strongly injective embedding, $\varphi : S^3 \longrightarrow SO(4)$, $\varphi(x) = h(x,\psi(x))$. It remains to show that $|d\psi| < 1$ implies φ is a smooth strongly injective embedding.

Suppose φ is not a smooth strongly injective embedding. Then for some $w \in S^3$, φ_w is not a diffeomorphism. Hence there is a point $p \in S^3$ and a unit vector $V \in U_pS^3$ (unit tangent space at p) such that $(d\varphi_w)_p(V) = 0$. By an argument completely analogous to that at the beginning of the "if" part of the proof, we may suppose $p = \psi(p) = 1$ and V is the unit tangent vector to the curve $\sigma(t) = \cos t + i \sin t$ at $t = 0$.

So we can apply equation 5.15.2, this time knowing $(d\varphi_w)_1(V) = 0$ to conclude $[\frac{d}{dt}\Big|_{t=0} H(t)](w) = 0$. By (5.15.1) we must have $[-1+\psi_2'(0)^2+\psi_3'(0)^2+\psi_4'(0)^2]^2 = 0$ hence $|(\psi\cdot\sigma)'(0)|^2 = \psi_2'(0)^2 + \psi_3'(0)^2 + \psi_4'(0)^2 = 1$ and $|d\psi| = 1$. QED.

THEOREM 5.16. Any great 3-sphere fibration of $S^3 \times S^3$ must contain some orthogonal pair of fibres.

PROOF: Corresponding to any great 3-sphere fibration of $S^3 \times S^3$ is a strongly injective embedding $\varphi : S^3 \longrightarrow SO(4)$. Corresponding to φ is a distance decreasing map

$\phi : S^3 \longrightarrow S^3$, mapping one factor of the double cover of SO(4) to the other, say WLOG, the first to the second.

ϕ distance decreasing implies $-\phi(x) \notin \mathrm{Im}\,\phi$ for any $x \in S^3$ so ϕ is not surjective. By the Borsuk-Ulam Theorem, there exist $\pm u$ in the domain of ϕ such that $\phi(u) = \phi(-u)$. Let $P_{\pm u}$ denote the fibres over $\pm u$,

$$P_{\pm u} = \{\frac{1}{\sqrt{2}}(v, \pm uv\phi(u)) : \ v \in S^3\}.$$

For $V = (v, uv\phi(u)) \in P_u$ and $W = (w, -uw\phi(u)) \in P_{-u}$ we compute $V \cdot W$. Note that for quaternions, $a = a_1 + a_2 i + a_3 j + a_4 k$ and $b = b_1 + b_2 i + b_3 j + b_4 k$, $\mathrm{Re}(a\bar{b}) = a_1 b_1 + a_2 b_2 + a_3 b_3 + a_4 b_4 = a \cdot b$, hence

$$\begin{aligned} V \cdot W &= \mathrm{Re}(v\bar{w}) + \mathrm{Re}(uv\phi(u)(\overline{-uw\phi(u)})) = \\ &= \mathrm{Re}(v\bar{w} - uv\phi(u)\overline{\phi(u)}\,\bar{w}\bar{u}) = \mathrm{Re}(v\bar{w} - uv\bar{w}\bar{u}) = \\ &= \mathrm{Re}(v\bar{w}) - \mathrm{Re}(uv\bar{w}u^{-1}) = \mathrm{Re}(v\bar{w}) - \mathrm{Re}(v\bar{w}) = 0 \end{aligned}$$

(recall that conjugation by a norm one quaternion fixes the real axis hence $\mathrm{Re}(uv\bar{w}u^{-1}) = \mathrm{Re}(v\bar{w})$).

Therefore we have shown $P_u = P_{-u}{}^{\perp}$. QED.

From these three theorems we see a very strong analogy with great circle fibrations of S^3 and the work of [G-W]. To keep this analogy going we would like to distinguish a certain "nice" subspace of the space of all great 3-sphere fibrations of $S^3 \times S^3$ and call them Hopf fibrations.

If we take the Hopf fibration of S^7 by great 3-spheres as the graphs of left multiplication in $\mathbb{H}$ (described in Section IV) and restrict to graphs of norm one quaternions, then clearly these 3-spheres fibre $S^3 \times S^3$. This fibration corresponds to the distance decreasing map $\phi : S^3 \longrightarrow 1 \in S^3$ (first factor to the second). Certainly this fibration of $S^3 \times S^3$ should be called a Hopf fibration. As in [G-W], any orthogonal transformation of this fibration (which still fibre $S^3 \times S^3$) should also be called a Hopf fibration. Those orthogonal transformations which fix $S^3 \times S^3$ are of the form $\begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix}$ where $A_i \in O(4)$.

Restricting to the special orthogonal group, so we stay in the class of strongly injective embeddings of S^3 in $SO(4)$ suppose first that $A_i \in SO(4)$ for $i = 1$ and 2. In this case we can represent the action of such a transformation as

$$(a,b) \longrightarrow (xay,ubv) \quad \text{for} \quad (a,b) \in S^3 \times S^3 \subseteq S^7$$

where x , y , u and v are elements of S^3 .

If $P_r = \{\frac{1}{\sqrt{2}}(c,rc): \ c \in S^3\}$ is a great 3-sphere of our Hopf fibration, then under the transformation given above

$$P_r \longrightarrow \{\frac{1}{\sqrt{2}}(xcy,urcv): \ c \in S^3\}$$

$$= \{\frac{1}{\sqrt{2}}(c',(urx^{-1})c'(y^{-1}v)): \ c' \in S^3\}$$

So the distance decreasing map corresponding to this new fibration is $\psi'(w) = y^{-1}v = \text{constant}$.

Suppose now $A_i \in O(4) - SO(4)$ for $i = 1$ and 2 . We get

$$(a^1,a^2) \longrightarrow (A_1a^1,A_2a^2) = (A_1{}'\bar{a}^1,A_2{}'\bar{a}^2)$$

where if $a^i = (a_1{}^i,a_2{}^i,a_3{}^i,a_4{}^i)$ then $\bar{a}^i = (-a_1{}^i,a_2{}^i,a_3{}^i,a_4{}^i)$, $i = 1$ and 2 , and $A_i{}' \in SO(4)$, $i = 1$ and 2 . So nothing new happens in this case and we conclude: Any special orthogonal transformation of the Hopf fibration of $S^3 \times S^3$, which still fibres $S^3 \times S^3$, has a corresponding distance decreasing map of the form

$\psi(S^3)$ = constant. Clearly all of this could have been applied to the "other" Hopf fibration of S^7 given by graphs of right quaternion multiplication so the map $\psi(S^3)$ = constant can be a map from either S^3 factor to the other.

DEFINITION 5.17. A Hopf fibration of $S^3 \times S^3$ is any great 3-sphere fibration of $S^3 \times S^3$, which induces a strongly injective embedding $\varphi : S^3 \longrightarrow S^3 \times S^3$ such that $\mathrm{Im}\ \varphi = (S^3, pt)$ or (pt, S^3).

THEOREM B. The space of all oriented great 3-sphere fibrations of $S^3 \times S^3 \subseteq S^7$ deformation retracts to the subspace of Hopf fibrations and hence has the homotopy type of a disjoint union of four copies of $\mathbb{R}P^3$.

PROOF: Let $\mathcal{DD}\mathfrak{m}(S^n)$ be the space of distance decreasing map of S^n to itself. We give $\mathcal{DD}\mathfrak{m}(S^n)$ the compact open or C^0 topology. Two maps f and g from S^n to itself are within ϵ of each other provided $f(x)$ and $g(x)$ are within ϵ of each other for all $x \in S^n$. For $f \in \mathcal{DD}\mathfrak{m}(S^n)$, f distance decreasing implies $-f(x) \notin \mathrm{Im}\ f$ for all $x \in S^n$ so f is not surjective. By the Borsuk-Ulam Theorem, there are $u \in S^n$ with $f(u) = f(-u) = u'$.

For all $x \in S^n$, $\min d(x,\pm u) \leq \frac{\pi}{2}$, so f distance decreasing implies $d(f(x),u') < \frac{\pi}{2}$ hence Image $f \subseteq$ open hemisphere of S^n.

LEMMA 5.18. There is a continuous map $c : \mathcal{DD}m(S^n) \longrightarrow S^n$ such that for each $f \in \mathcal{DD}m(S^n)$, the image $f(S^n)$ lies in the open hemisphere centered at $c(f)$.

PROOF: Im f certainly vaires continuously with f by the choice of topology on $\mathcal{DD}m(S^n)$. Let $B(f)$ be the closed ball of smallest radius which contains the closed set Im f.

1) $B(f)$ is uniquely determined by f.

This follows from the fact that on the unit n-sphere the intersection of two closed balls, each of radius $< \pi/2$ is contained in some closed ball of smaller radius.

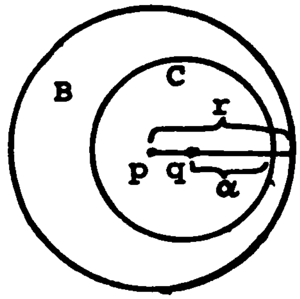
Let $c(f)$ denote the center of the ball $B(f)$ and $r(f)$ its radius.

2) $r(f)$ varies continuously with f.

If f is perturbed by less than ϵ to g then $r(g) < r(f) + \epsilon$ and by symmetry $r(f) < r(g) + \epsilon$.

3) Let B be a ball of radius $r < \pi$ on S^n with center p then for any ball C of radius α, $\alpha > r-\beta$, inside B with center q, $d(p,q) < \beta$.

This follows easily since if $p \neq q$, then the great circle segment from p to q to ∂B has length r. The portion of the segment from q to ∂B has length $\geq \alpha > r-\beta$, so the portion from p to q has length $< \beta$.



Now for $\pi/2 > \epsilon > 0$ given, let $\mathcal{U}_\epsilon$ be the open ball about f, $\mathcal{U}_\epsilon = \{g \in \mathfrak{D}\mathfrak{D}\mathfrak{m}(S^n)\colon$ distance from f to g is less than $\epsilon/2\}$. Let B be the ball of radius $r(f)+\epsilon/2$ with center $c(f)$, then Image $g \subseteq B$ for all $g \in \mathcal{U}_\epsilon$, hence $B(g) \subseteq B$ for all $g \in \mathcal{U}_\epsilon$. Now (2) implies $r(f) - \epsilon/2 < r(g)$ so $(r(f)+\epsilon/2) - \epsilon < r(g)$ and by (3) we conclude $d(c(f),c(g)) < \epsilon$. Since we can find such a $\mathcal{U}_\epsilon$ for all $\epsilon > 0$ this implies c is continuous.

QED LEMMA.

Now we set $n = 3$. For any $f \in \mathfrak{D}\mathfrak{D}\mathfrak{m}(S^3)$, radial contraction of $\mathrm{Im}(f)$ to $c(f)$ homotopes f, through distance decreasing maps to a constant map, $f_0\colon S^3 \longrightarrow c(f) \subseteq S^3$. By LEMMA 5.18, $c(f)$ depends continuously on f so this process is a deformation retraction hence $\mathfrak{D}\mathfrak{D}\mathfrak{m}(S^3)$ has the homotopy type of S^3.

Since $S^3 \times S^3$ is the double cover of $SO(4)$ the great 3-sphere fibrations of $S^3 \times S^3$ determined by the two constant maps, $f_1: S^3 \longrightarrow p$ and $f_2: S^3 \longrightarrow -p$ both from the first factor to the second or vice versa, are identical. So the family of great 3-sphere fibrations of $S^3 \times S^3$ determined by strongly injective embeddings $\varphi : S^3 \longrightarrow SO(4)$ has the homotopy type of a *disjoint* union of two copies of $\mathbb{R}P^3$. We get two more copies of $\mathbb{R}P^3$ by considering strongly injective embeddings $\varphi : S^3 \longrightarrow O(4) - SO(4)$. QED.

Except for the last paragraph and part (3) of the LEMMA, the proof of THEOREM B is essentially identical to ([G-W], Theorem D).

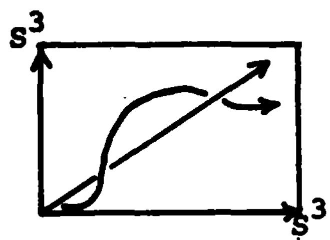
With this theorem we have answered completely, for great 3-sphere fibrations of $S^3 \times S^3$ the three questions posed on page 1:

1) Are all such fibrations topologically equivalent? Yes, they are all trivial.

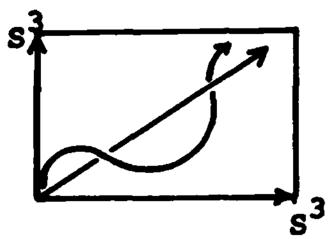
2) Given two equivalent fibrations, is it possible to deform one to the other, through the space of great 3-sphere fibrations of $S^3 \times S^3$? Not necessarily, there are four distinct deformation

classes depending on whether the base space embeds in SO(4) or O(4) - SO(4) and as the graph of a distance decreasing map from the first factor to the second or vice versa.

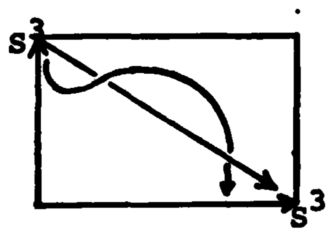
Pictorially we may display these four deformation classes as follows:



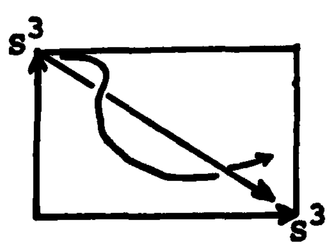
Base space embeds in SO(4), lifts to $S^3 \times S^3$ as $(1, S^3)$



Base space embeds in SO(4), lifts to $S^3 \times S^3$ as $(S^3, 1)$



Base space embeds in O(4)-SO(4), lifts to $S^3 \times S^3$ as $(1, S^3)$



Base space embeds in O(4)-SO(4), lifts to $S^3 \times S^3$ as $(S^3, 1)$

3) What is the homotopy type of the space of all such fibrations? By THEOREM B, the homotopy type is four copies of $\mathbb{R}P^3$.

Given a great 3-sphere fibration of S^7 , $F : S^3 \longrightarrow S^7 \longrightarrow S^4$, that also fibres $S^3 \times S^3$, we may take for the classifying map of the bundle F , the map of the base space of the fibration restricted to $S^3 \times S^3$ into $O(4)$. By THEOREM B, the homotopy type of such a strongly injective embedding of S^3 in $SO(4)$ is $(0,1)$ or $(1,0)$ hence we conclude that all such fibrations F are topologically equivalent to the Hopf fibration.

Now we pose the question: Can every great 3-sphere fibration of $S^3 \times S^3$ appear as a portion of a great 3-sphere fibration of S^7 ?

We will see the answer is yes. Let $P_0 = \mathrm{span}(e_1,e_2,e_3,e_4) \subseteq \mathbb{R}^8$, and $P_\infty = \mathrm{span}(e_5,e_6,e_7,e_8) \subseteq \mathbb{R}^8$. Our plan is to combine two relatively simple concepts using the deformation provided in THEOREM B.

1) Every great 3-sphere fibration of $S^3 \times S^3$ can be "fattened up". Suppose $\phi : S^3 \longrightarrow S^3$ is a distance decreasing map giving a fibration of $S^3 \times S^3$. It is a

simple matter to check that the family of 4-planes

$$\{P_{ta}: \ a \in S^3 , \ 0 < t < \infty\} \quad \text{where}$$

$$P_{ta} = \{(u, tau\psi(a)) : \ u \in \mathbb{H}\}$$

gives a fibration of $S^7 - (P_0 \cup P_\infty)$.

Viewed from the perspective of strongly injective embeddings, we have given a map $\varphi : S^3 \longrightarrow O(4) \subseteq GL(4,\mathbb{R}) \cup \{0\} \subseteq Hom(\mathbb{R}^4,\mathbb{R}^4)$ where φ is a strongly injective embedding. Using the linear structure of $Hom(\mathbb{R}^4,\mathbb{R}^4)$, we extend φ to a map $\tilde{\varphi} : \mathbb{R}^4-\{0\} \longrightarrow GL(4,\mathbb{R})$ via $\tilde{\varphi}(x) = \|x\|\varphi(\frac{x}{\|x\|})$. One can readily confirm that φ a (smooth) strongly injective embedding implies $\tilde{\varphi}$ is a (smooth) strongly injective embedding. Clearly the graphs of the image of $\tilde{\varphi}$ are just the family of 4-planes above.

This viewpoint also illustrates that if we include the 4-plane P_0 (extend φ with $\varphi(0) = 0$) then while we have a topological fibration it is no longer necessarily C^∞. Of course an analogous problem occurs at P_∞.

2) If $\psi : S^3 \longrightarrow$ constant, $c \in S^3$, then we can include P_0 and P_∞ and retain differentiability. If $c = 1$ this is clear for then the fibration obtained by

"fattening up" as in (1) are just the regular Hopf fibrations (graphs of left or right quaternion multiplication). But in the discussion preceding definition 5.17 we see that those fibrations determined by $\psi : S^3 \longrightarrow c$ are given by orthogonal transformation, A , applied to $\mathbb{R}^8$ that fixes $S^3 \times S^3$ and consequently P_0 and P_∞ . Its clear that the result of "fattening up" the fibration $\psi : S^3 \longrightarrow c$ gives the same fibration of S^7 as A applied to the regular Hopf fibration.

THEOREM C. Every (smooth) great 3-sphere fibration of $S^3 \times S^3$ can be extended to a (smooth) great 3-sphere fibration of S^7 .

PROOF: Given a great 3-sphere fibration of $S^3 \times S^3$, let $\psi : S^3 \longrightarrow S^3$, as usual, be the corresponding distance decreasing map. Let $\psi_t : S^3 \longrightarrow S^3$, $t \in I$, be the homotopy of ψ , through distance decreasing maps, to a constant map provided by THEOREM B, with $\psi_1 = \psi$ and $\psi_0(x) = c$ for all $x \in S^3$.

Let $\eta : [0,\infty) \longrightarrow [0,1]$ be a C^∞-function such that $\eta(1) = 1$ and support $\eta \subseteq (1/2,2)$. We suppose $\mathbb{R}^8 = \mathbb{H} \times \mathbb{H}$ and consider the family of 4-planes:

$\{P_0\} \cup \{P_\infty\} \cup \{P_{tv}: \ 0 < t < \infty\}$ where

$P_{tv} = \{(u, tvu\phi_{\mathcal{H}(t)}(v)): \ u \in \mathbb{H}\}$.

Note that for $t < 1/2$ and $t > 2$, $\phi_{(t)}$ is a constant map so the fibration is smooth in a neighborhood of P_0 and P_∞ (by (2) above). With this it is easy to verify that the above family fibres S^7 and the fibration is smooth if the original fibration of $S^3 \times S^3$ is smooth. QED.

As a final application of THEOREM 5.13 we re-prove the well known fact that the Grassman manifold of oriented 2-planes in $\mathbb{R}^4$, $\tilde{G}_2\mathbb{R}^4$, is homeomorphic to $S^2 \times S^2$. From this and some related facts it can be seen that Theorems A-D of [G-W] are essentially corollaries of our THEOREMS 5-14, 15, 16 and THEOREM B.

Let $S^2 \times S^2 \subseteq S^3 \times S^3$ be the set $(x,y) \in S^3 \times S^3$ with $\mathrm{Re}\, x = \mathrm{Re}\, y = 0$. Let $P = \{(u,u): \ u \in \mathbb{H}\} \subseteq \mathbb{R}^8$. P is a 4-plane and P is the graph of $h(1,1) = \mathrm{id} \in SO(4)$ as a map from P_0 to P_∞. In general we write $P_{(x,y)}$ for the 4-plane given by the graph of $h(x,y)$, so $P = P_{(1,1)}$.

LEMMA 5.18. $R_{(x,y)} = P \cap P_{(x,y)}$ is a 2-plane in P for all $(x,y) \in S^2 \times S^2$.

PROOF: $(x,y) \in S^2 \times S^2$ implies $d(x,1) = d(y,1)$ so by Theorem 5.13, $h(x,y)$ has a fixed point $w \in S^3$. But the fixed point set of an element of SO(4) that is not the identity is either 0 or 2 dimensional, hence $h(x,y)$ has a 2-dimensional fixed point set. Since $P \cap P_{(x,y)}$ is the fixed point set of $h(x,y)$ we conclude $R_{(x,y)}$ is a 2-plane. QED

So if we identify P with $\mathbb{R}^4$ we see that $(x,y) \in S^2 \times S^2$ corresponds to a 2-plane $R_{(x,y)}$ in P. Now we'd like to show that every 2-plane in P arises in such a manner. This will follow from

LEMMA 5.19. For any $x,y \in S^3$, $y \in x^{\perp}$, the system of equations $uxv = x$ and $uyv = y$ has a solution $(u,v) \in S^2 \times S^2$ and the solution is unique up to sign.

PROOF: $y \in x^{\perp}$ implies $y \neq \pm x$ so $yx^{-1} \neq \pm 1$. So there is a unique great circle through 1 and yx^{-1} intersecting the purely imaginary 2-sphere in antipodal points $\pm u$. This great circle contains all elements of S^3 that commute with yx^{-1} so $uyx^{-1}u^{-1} = yx^{-1}$. Consequently, $uy(x^{-1}u^{-1}x) = y$. Now conjugation fixes the purely imaginary 2-sphere so $u \in S^2$ implies $u^{-1} = -u \in S^2$ and hence $x^{-1}u^{-1}x = v \in S^2$. Therefore $uyv = y$ and $uxv = x$.

Now $\pm u$ are the only points of S^2 that commute with yx^{-1} . If $u'xv' = x$ and $u'yv' = y$ then $y'yx^{-1}u'^{-1} = yx^{-1}$ so u' commutes with yx^{-1} hence $u' = \pm u$. Therefore our solution is unique up to sign. QED.

LEMMA 5.19 tells us that for any pair of orthogonal vectors in S^3 , there exists $(x,y) \in S^2 \times S^2$ such that the 2-plane spanned by the orthogonal pair is precisely the fixed point set of the orthogonal transformation $h(x,y)$. Consequently every 2-plane in P arises as $P \cap R_{(x,y)}$ for some $(x,y) \in S^2 \times S^2$. Because of the uniqueness statement in the LEMMA we get a bijective correspondence between 2-planes in P and points of $S^2 \times S^2/(x,y) = (-x,-y)$. If we choose an orientation in the 2-plane $R_{(i,i)}$ say, and extend continuously, we get a bijective correspondence between oriented 2-planes in P and points in $S^2 \times S^2$.

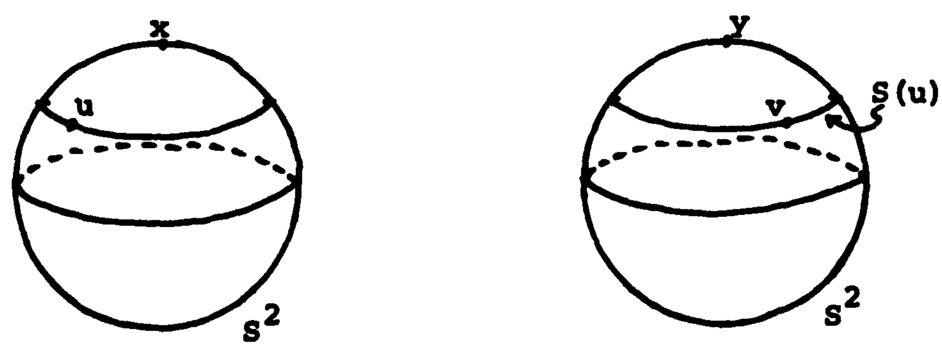
Finally, we'd like to show how the base space of a great circle fibration of $S^3 \subsetneq P$ appears in $S^2 \times S^2$. From THEOREM 5.13, if $R_{(x,y)}$ intersects $R_{(u,v)}$ in more than the origin then we know $d(x,u) = d(y,v)$. A priori, however, if $d(x,u) = d(y,v)$, then while we know $P_{(x,y)}$ intersects $P_{(u,v)}$ in more than the origin, its not clear that this intersection need appear inside P. The next lemma says that this does happen.

LEMMA 5.20. $R_{(x,y)}$ intersects $R_{(u,v)}$ in more than the origin, if and only if $d(x,u) = d(y,v)$ in $S^2 \times S^2$.

PROOF: ($\Rightarrow$) THEOREM 5.13.

($\Leftarrow$) Let $S = \{w \in S^3 : h(x,y)(w) = xwy = w\}$. From LEMMA 5.18 we conclude S is a great circle since S is the fixed point set in S^3 of $h(x,y) \in SO(4)$. For $u \in S^2$, $u \neq \pm x$ (where the result is clear), let

$$S(u) = \{z \in S^2 : d(x,u) = d(y,z)\}$$



$S(U)$ is homeomorphic to S^1. If

$$F(u) = \{z \in S^2: \ h(u,z)(w) = w \ \text{ for some } \ w \in S\}$$

then by THEOREM 5.13 we conclude $F(u) \subseteq S(u)$.

But $h(u,z)(w) = w \Leftrightarrow uwz = w \Leftrightarrow z = w^{-1}u^{-1}w$ so we have a map $\tilde{\tau} : S \longrightarrow F(u)$, $\tilde{\tau}(w) = w^{-1}u^{-1}w$. If we factor $\tilde{\tau}$ through $\mathbb{R}P^1 = \{[w]: [w] = \pm w \text{ for } w \in S\}$ then we get a map $\tau : \mathbb{R}P^1 \longrightarrow F(u)$. τ must be injective since if $w_1^{-1}uw_1 = w_2^{-1}uw_2 = z'$ for $w_1 \neq \pm w_2$ then w_1 and w_2 span S and $uwz' = w$ for all $w \in S$. This implies $R_{(x,y)} = \{(S,S)\} = R_{(u,z')}$ so by the uniqueness part of LEMMA 5.19, $(u,z') = \pm(x,y)$ and $u = \pm x$, a contradiction. Therefore τ is a continuous injection and $F(u)$ is homeomorphic to $\mathbb{R}P^1 \simeq S^1$, contained in $S(u) \simeq S^1$. This can happen only if $F(u) = S(u)$. Since $v \in S(u)$, we conclude $v \in F(u)$ and there is a $w \in S$ with $h(u,v)(w) = uwv = w = xwy = h(x,y)(w)$ so $(w,w) \in R_{(x,y)} \cap R_{(u,v)}$. QED.

Thus we have established the connection between great circle fibrations of S^3 and distance decreasing maps $\phi : S^2 \longrightarrow S^2$.

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